TOWARDS EFFECTIVE MATHEMATICS PEDAGOGY IN INDIA

INSIGHTS FROM RESEARCH

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To my children Chetna and Rajiv

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Preface

Mathematics is a fundamental life skill essential for daily living, engagement with the world, cognitive growth, and problem-solving. Yet in India, far too many children experience mathematics not as an empowering discipline but as a barrier to academic and personal success.

Despite multiple national policies and programmes, persistent challenges remain in achieving foundational numeracy, developing conceptual understanding, problem-solving, and higher-order thinking in maths, and improving examinations.

Policymakers, educational leaders, teacher educators, and teachers generally overlook evidence-based strategies for improving education. Fads and ideology frequently hold more sway than scientific research.

This book responds to that reality. It aims to provide an integrated, evidence-based approach to enhancing mathematics pedagogy in the Indian context based on national datasets, drawing on international meta-analyses and research in cognitive psychology and neuroscience.

The intended audience includes policymakers, curriculum developers, textbook writers, educational leaders, teacher educators, mathematics teachers, and researchers. Each chapter is designed to connect theory with practice.

We hope the stakeholders will consider the ideas presented here and incorporate what resonates with them in their policy, curriculum, textbooks, training of pre-service and professional development of in-service teachers, and teaching to improve maths education and evaluate them.

Introduction

Mathematics is more than a school subject—it is a vital life skill. It enables people to manage everyday tasks, understand patterns in the world, think critically, make informed decisions, and contribute to innovation and development. Strong mathematics education also underpins learning in many other subjects, employability and a nation's competitiveness in a rapidly changing world.

In India, where education is central to shaping the lives of millions, the teaching and learning of mathematics is especially significant. It is compulsory until class 10, yet for many students, mathematics remains a source of anxiety and disengagement.

Persistent gaps in foundational numeracy and overall achievement have been documented by successive national surveys, including The Annual Status of Education Reports (ASER), the National Achievement Surveys (NCERT), and the PARAKH Rashtriya Sarvekshan (NCERT). These consistently show that a large proportion of learners struggle with basic numeracy, conceptual understanding, and problem-solving.

The National Education Policy (2020) emphasises the urgent need to strengthen mathematics teaching. It identifies persistent challenges such as weak foundational skills, an overemphasis on rote memorisation over conceptual understanding and higher-order thinking, limited links between mathematics and everyday life, widespread math anxiety, and a shortage of qualified teachers. The policy also calls for reform of board examinations, which currently emphasise rote memory, procedures, and routine applications, thereby shaping classroom practices in unproductive ways.

This book is a response to deep concern for these persistent issues and from a belief in the transformative potential of research-informed teaching. It synthesises insights from national and international research in

education, cognitive psychology and neuroscience to provide a comprehensive yet practical guide for policymakers, educational leaders, curriculum developers, teacher educators, mathematics teachers, and others committed to making mathematics learning meaningful and effective.

Purpose of the Book

The central aim is to ground mathematics teaching practices in research. By synthesising research in education, cognitive psychology and neuroscience, this book translates research into actionable strategies for the classroom. Findings from international meta-analyses have been mainly used, as findings from individual studies depend to a large degree on the context. (Meta-analysis is a statistical method used to systematically combine and analyze the results of multiple independent studies on the same topic to draw an overall conclusion.) The goal is to support educators in achieving foundational numeracy and foster conceptual understanding, communication skills, problem solving and higher-order thinking, and develop traits that promote learning, e.g. curiosity, motivation, growth mindset, perseverance, and self-efficacy, and tackle ones that hinder learning, e.g. anxiety, procrastination.

While meta-analyses provide broad insights, their effectiveness depends on implementation and specific contexts. Therefore, approaches should always be adapted and evaluated locally. The scope of this book is limited to mathematics education in Classes 1 to 10.

Who is This Book for?

This book is intended for policymakers, curriculum developers, textbook writers, educational leaders, teacher educators, mathematics teachers, researchers, and others interested in strengthening mathematics pedagogy in India

Structure of the Book

Chapter 1: Reviews the current landscape of mathematics education in India, including policy, curriculum, national surveys, teacher quality, and government and non-government initiatives to improve maths education.

Chapter 2: Explores findings from John Hattie's meta-analyses (2023), synthesising effects of over 350 variables across 130,000 studies and 400 million students.

Chapter 3: Provides teaching strategies suggested by research in cognitive psychology, viz. engagement, retrieval, spaced practice, and interleaving that enhance learning and key affective traits that influence learning.

Chapter 4: Describes how the brain learns maths and its implications for teaching.

Chapter 5: Outlines the goals of mathematics education – conceptual understanding, fluency of procedures, communication skills, and higher-order thinking – and suggests evidence-based strategies to achieve them.

Chapter 6: Discusses the importance of formative assessment for learning, tools to conduct it, and its challenges. And the importance and characteristics of effective feedback for learning.

Chapter 7: Examines the need for professional development for pre-service and in-service teachers, characteristics of effective professional development that may be provided to teachers and online resources available to teachers.

Chapter 8: Concludes with reflections.

Limitations of the Book

This book has not addressed integration of digital tools, as most students in India do not have access to a computer in school or at home and some other issues such as teaching maths to students whose mother tongue is different from the medium of instruction in school, social-emotional learning.

Some meta-analyses on maths education may have been missed.

A Call for Collective Action

While teachers play a central role in shaping students' learning experiences, meaningful and lasting improvements in mathematics education require coordinated efforts from a wide range of stakeholders. Policymakers must create enabling environments through supportive policies and adequate resources. Curriculum developers and textbook writers must design learning and assessment materials that are relevant, coherent, and aligned with both students' developmental needs and the goals of the National Education Policy. Teacher educators need to equip future teachers with pedagogical and deeper understanding of content based on research and an inquiry-orientated mindset. Researchers should provide evidence on what works in different contexts. While parents and communities can help build positive attitudes towards mathematics and support learning beyond school walls.

This book is intended to serve as a bridge between research and practice. It is hoped that the insights presented within these pages will spark informed dialogue, encourage thoughtful reflection among educators and decision-makers, and promote the adoption of practices rooted in empirical research. By working together, we can make mathematics learning more engaging and meaningful and ensure that all learners, regardless of background, have the opportunity to develop mathematical thinking that empowers them in school and in life.

1

The Current Landscape of Mathematics Education in India

With a population of more than 143 crore (1.43 billion), it’s no surprise that India runs the largest national school system in the world with lots of challenges.

India has made great progress in increasing enrolment at the elementary level (classes 1 to 8) due to the Sarva Shiksha Abhiyan programme (SSA) and the Right to Free and Compulsory Education Act (2009). However, the attainment of foundational literacy and numeracy and the quality of maths education are of great concern.

We will concern ourselves with maths education for classes 1 to 10, which is compulsory for all students, and consider national data only. The data on different states, districts, and schools within a district generally differ significantly.

The Number of Students, Teachers, Schools, and Average Pupil-Teacher Ratio

India's education system at the school level is broadly divided into the following stages:

1. Pre-Primary Stage: Ages 3-6.
2. Primary Stage: Ages 6-11 (Classes 1-5)
3. Upper Primary Stage: Ages 11-14 (Classes 6-8)
4. Secondary Stage: Ages 14-16 (Classes 9-10)
5. Higher Secondary Stage: Ages 16-18 (Classes 11-12).

For further discussion, we will consider *national data* for classes *1-10* only.

The number of students, teachers, schools, and pupil-teacher ratio in different stages of education in the year 2021-2022 for India according to the UDISE+ report are given in Table 1.1.

Table 1.1: The number of students, teachers, schools, and pupil-teacher ratio in different stages of education in 2021-2022

Stage	Number of students	Number of teachers	Number of schools	Pupil-teacher ratio
Primary	12,18,42,250	24,33,800	7,60,460	26
Upper Primary	6,67,90,692	3,50,374	4,35,374	19
Secondary	3,85,28,631	1,72,028	1,50,452	18

The Gross Enrolment Ratio in Different Stages of Education in India

The growth in enrolment ratios (GER) at various stages

There has been a steady increase in GER at various levels, indicating a growing number of children attending school (UDISE+, 2021). The improvements in GER from 2018-2019 to 2021-2022 in various stages is given in Table 1.2.

Table 1.2: The growth in gross enrollment ratio (GER) from the years 2018-2019 to 2021-2022 (UDISE+, 2021)

Stage	GER in 2018-2019	GER in 2021-2022
Primary	101.3%	104.8%
Upper Primary	87.74%	94.7%
Secondary	76.9%	79.6%

There is a need to collect data on the extent of absenteeism.

Dropout Rates

The Unified District Information System for Education Plus (UDESI+) – a database about schools in India – reported the following rate of dropouts for various stages of education in 2021-2022:

Primary Level (Classes 1-5): The dropout rate is the lowest at this level, with a national average of approximately 1.5%.

Upper Primary Level (Classes 6-8): The dropout rate increases slightly to an average of around 3%.

Secondary Level (Classes 9 & 10): At this level, the dropout rate increases significantly with a national average of approximately 12.6%.

Gender disparity in dropout out rates

Dropout rates are generally higher for girls compared to boys at all levels of education.

Socioeconomic background disparity in dropout out rates

Students from disadvantaged backgrounds are more likely to drop out of school due to financial constraints or the need to contribute to the family income.

Attainment of Foundational Literacy and Numeracy

According to the 86 Constitutional Amendment Act, 2002, free and compulsory education for all children in the 6-14 year age group is now a fundamental right under Article 21-A of the constitution.

Basic foundational skills like reading and arithmetic are necessary for daily activities like shopping, cooking, understanding instructions, budgeting, filling out forms, using online services for paying bills, or booking railway or airline tickets.

The Annual Status of Education Report (ASER) (2023) survey was conducted in 28 districts across 26 states with a total of 34,745 youth in the age group 14-18 years. One rural district was surveyed in each major state, except Uttar Pradesh and Madhya Pradesh, where two rural districts were surveyed.

Data was collected regarding youths' current activity and their ability to do basic tasks and applications to daily life. The youths' access to digital devices and skills to do digital tasks were also investigated.

A sample of the findings is given below:

- About 25% of this age group could not read a Std II level text fluently in their regional language.
- More than half struggled with division (3-digit by 1-digit) problems. Only 43.3% of 14- to 18-year-olds could do such problems. More than 60% could do a budget task, about 37% could apply a discount, but only about 10% could calculate repayment. About 50% of the youth could do other common calculations, such as calculating time, adding weights and applying the unitary method.

- A little over half could read sentences in English (57.3%). Of those who could read sentences in English, 73.5% could tell their meanings.

- Females (76%) did better than males (70.9%) in reading a Std II level text in their regional language. In contrast, males did better than their female counterparts in arithmetic and English reading.

A study by Barros, A.D. & Ganimian, A.J. (2023) indicated that while students perform poorly in the basic skills regularly assessed by ASER, they perform worse on foundational skills in fractions, decimals, measurement and geometry which are not assessed by ASER but are important for further learning. In their study based on 101,084 public-school students in grades 4, 6, and 8 across 19 Indian states, only 27% of the students were proficient in the skills not assessed by ASER vs 52% in the skills assessed by ASER. They further noted that while student achievement in previously assessed skills improves across grades from 43% in grade 4, to 50% in grade 6, to 61% in grade 8, the corresponding trajectory in skills not assessed did not improve much. It was 22% in grade 4, 28% in grade 6, and 29% in grade 8. And girls perform below boys in these skills. The gap emerges between grades 4 and 6 and remains unchanged in grade 8. In grade 4, the percentage of boys and girls who have mastered these skills was 22%; by grade 6, 25% of girls and 31% of boys had done so, and this gap persisted in grade 8.

These data show that the development of numeracy needs substantial improvement. We also need to develop foundational skills in fractions, decimals, measurement and geometry important for further learning in maths.

Digital access and use(self-reported)

- Close to 90% of all youth had a smartphone in their household and knew how to use it. Of those who could use a smartphone, males (43.7%) were more than twice as likely to have a smartphone as females (19.8%).

- The availability of a computer/laptop in the households was much lower, with only 9% having one at home. Youth who had a computer/laptop at

home were much more likely to know how to use it (85%) than those who did not (33.9%).

- Females were less likely to know how to use a smartphone or computer than males.
- Half of all surveyed males had an email ID, compared to slightly under 30% of females. Among enrolled youth, the likelihood of having an email ID and having sent an email increased as the level of education went up.
- Almost all youth (90.5%) reported having used social media in the reference week, with a slightly higher proportion of males (93.4%) than females (87.8%).

Use of smartphones for learning

Among the youth who can use a smartphone, two-thirds reported having used it for some education-related activity during the reference week, such as watching online videos related to studies, clearing doubts, or exchanging notes. A quarter of the youth who were not currently enrolled in schools also reported doing education-related activities on their smartphones during the reference week.

Services and entertainment

- Slightly over a quarter of all youth reported having used a smartphone to access online services such as making online payments, filling out a form, paying a bill or booking a ticket.
- Males were more likely to have accessed at least one of these services (37.6%) than females (19%).
- Close to 80% of the youth reported having used their smartphone to do an entertainment-related activity, such as watching a movie or listening to music, during the reference week.

Online safety

Of the youth who used social media, only about half were familiar with the online safety settings that were included in the survey. Males were more likely to know about these settings than females.

Update on access to technology

Parakh's Rashtriya Suevekshan (2024) has provided data on access to technology in schools and at home.

Access to technology in schools

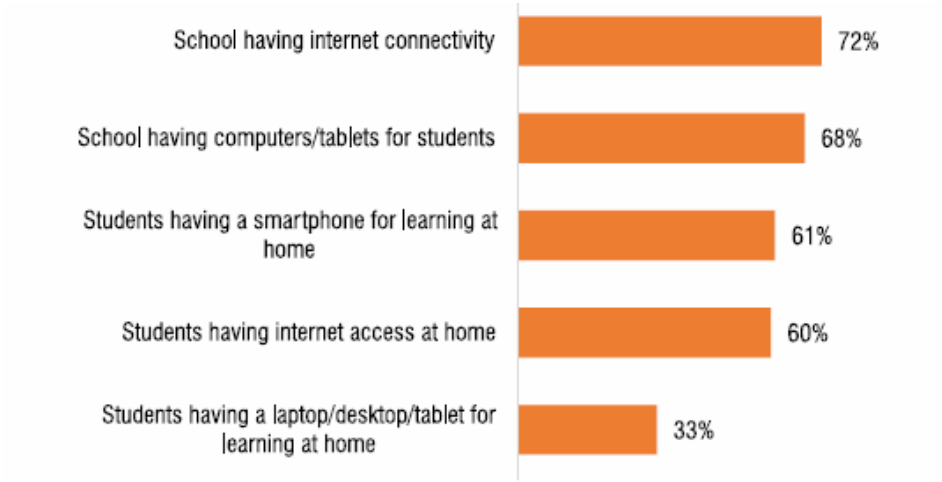


Figure 1.1

While access to smartphones and the internet at home has increased, the availability of laptops, desktops, or tablets for learning is low. There is a need to provide access to personal learning devices for all students to support personalised, interactive, inclusive learning and engagement of students and empower teachers.

There is also a need to collect data on the use of technology in schools.

Performance in Mathematics

A National Achievement Survey (NAS) in India was conducted by the National Council of Education Research and Development (NCERT) (2021)

for classes III, V, VIII, and X. The performance of students was classified into Below Basic, Basic, Proficient, and Advanced categories.

These performance levels for any grade and subject were described by the National Achievement Survey (2021) as follows:

Below basic	Basic	Proficient	Advanced
• Learners at this level are at the early stages of development regarding the curriculum standards. • They have not achieved sufficient knowledge and skills to be considered minimally successful regarding curriculum demands. • They need guidance at every stage of learning. They can make little judgement and need a lot of encouragement and guidance.	• Learners at this level demonstrate a minimum level of skills related to the curriculum learning outcomes. • They can follow simple instructions and apply simple rules to achieve expected performance. • They have some good ideas which often lack coherence. They need guidance at many stages of learning. • They can solve problems using simple logic, and can also express themselves using simple language.	• Learners at this level have acquired most of the learning outcomes and skills required by the curriculum. They can work independently with minimum supervision. They have a systematic methodology to solve problems. • They can communicate their ideas clearly. They can also connect different ideas and create meaning with minimum guidance and supervision. • They can analyze situations and interpret information for application to new situations.	• Learners at this level display exceptional mastery of the learning content as prescribed by the curriculum and beyond. They are independent with high analytical, reflective and critical thinking. • They can connect and integrate concepts and ideas to create new knowledge/ meaning and solve complex problems. • They communicate information with the highest level of creativity and coherence as well as make sound judgements.

The percentage of students in different categories of performance for classes 3, 5, 8, and 10 is given in Table 1.3.

Table 1.3: The percentage of students in different categories of performance for classes 3, 5, 8, and 10 (NAS (2021).

Class	Below Basic	Basic	Proficient	Advanced
3	23	34	30	12
5	34	41	19	6
8	27	46	19	8
10	30	47	17	6

The percentage of students in the proficient and advanced categories is very low and is a cause of great concern, especially for maths, which is a hierarchical subject.

Parakh (2024) conducted a national survey in which 21,15,022 children were assessed from grades 3, 6 and 9 from 74,229 schools from 781 districts across 36 states and UTs (State Govt., Govt. Aided, Private recognised and Central Govt. schools). The average scores of students are given in Table 1.4.

The low mastery of foundational skills by Class 3, along with the significant decline in learning outcomes from Class 3 to 6 and again from Class 6 to 9, highlights the urgent need to strengthen learning across all grade levels.

Table 1.4: The average scores of students in classes 3, 6 and 9 (Parakh's Rashtriya Suevekshan Survey (2024)).

Class	Average Score
3	60%
6	46%
9	37%

Qualifications of Mathematics Teachers

The qualification of teachers at the primary stage is a two-year course for teacher training after higher secondary (class 12) or Bachelor of Elementary Education. Most of the teachers have maths up to class 10 only. Teachers for the upper primary stage are required to have a B.A. or B.Sc. with maths and a B.Ed. degree.

Teachers for Classes 1 to 8 also have to pass a Teacher Eligibility Test (TET) conducted by the Central Board of Secondary Education (CTET) and state boards of secondary education. It has one paper for the primary and two papers for the upper primary stage. They have to get 60% to qualify.

The maths portion for the test is based on what is taught up to class 8 for elementary teachers and up to class 10 for upper primary teachers.

Availability of Mathematics Teachers

- In reply to a question in parliament in December 2023, the government mentioned a shortage of 7,22,413 teachers at the elementary level and 1,24,262 teachers at the secondary level.
- In a study in eight states conducted by Tata Institute of Social Sciences (TISS, 2023), in almost half the cases, both in government and private schools, teachers who have not studied mathematics in their undergraduate studies were teaching the subject in upper primary and secondary stages.

The Quality of Mathematics Teachers

The percentage of students who qualify in CTET and TETs conducted by the centre and states is very low. For example, results for CTET from (2015–2024) are given in Table 1.5 (ChatGPT)

Table 1.5: Performance in CTET examination from the year 2015 to 2024

Year	Appeared	Qualified	Overall Pass %
Jan 2024	~22.76 lakh	~2.39 lakh	~10.5%
Jul 2023	23.80 lakh	3.998 lakh	16.79%
Dec 2022	~26.99 lakh	9.558 lakh	~35.4%
Dec 2021	~23.52 lakh	6.54 lakh	~27.8%
Dec 2019	~24.00 lakh	5.42 lakh	~22.6%
Jul 2019	~23.77 lakh	3.52 lakh	~14.8%
Sep 2016	~6.53 lakh	0.07 lakh	~10.7%
Feb 2015	~6.78 lakh	0.80 lakh	~11.8%

Thus, it is essential to improve mathematics education and teacher training programmes to increase the number of candidates who can qualify to address the shortage of teachers.

Tata Institute of Social Sciences (2023), based on a study in eight states, about the quality of teachers reported

- Approximately 15% of qualifying candidates in TET secured more than 60% on the test, and about 1.5% scored above 70%.
- The scenario in government and government-aided schools is better, but in private schools, only about a fifth of primary school teachers have the right professional qualification.
- Among primary teachers, a large proportion of those qualifying to become teachers do not have adequate mathematics knowledge, as reflected in the mean marks of 46% in that component. The mean mark in mathematics content knowledge for the secondary science and mathematics teachers was 50% (TISS, 2023).

  Thus, there is a need to provide for deeper understanding of maths in teacher training programmes professional development of in-service teachers to improve teacher quality.

Teaching Practices

According to the draft of the National Curriculum Framework (2023), the traditional approach to teaching mathematics has been overly focused on algorithms and has neglected the subject's creative and aesthetic aspects.

The teaching methods have also relied too heavily on lectures, rote learning, and meaningless practice.

Status of Some Teaching Practices

The UDISE+ (2021) data on some teaching practices in government and government-aided schools is given in Tables 1.4.

Table 1.4: Percentage of government and government-aided schools using teaching-learning practices in India

Practice	Primary	Upper Primary	Secondary
Created tests based on learning outcomes	-	35.2	42.7
Undertook academic enrichment activities	-	26.0	29.7
Created criterion-referenced items	79.4	79.5	77.7
Created teaching aids/tools	73.0	77.3	79.2
Identified hard spots in learning outcomes	17.9	23.8	27.7
Oriented students on cyber safety	8.0	15.9	24.0
Trained students on psychosocial aspects	6.2	10.4	16.1
Introduced peer learning	51.9	51.8	50.4

It is crucial to expand the use of these practices and integrate more research-based approaches that have proven effective in enhancing mathematics education.

Assessment Practices

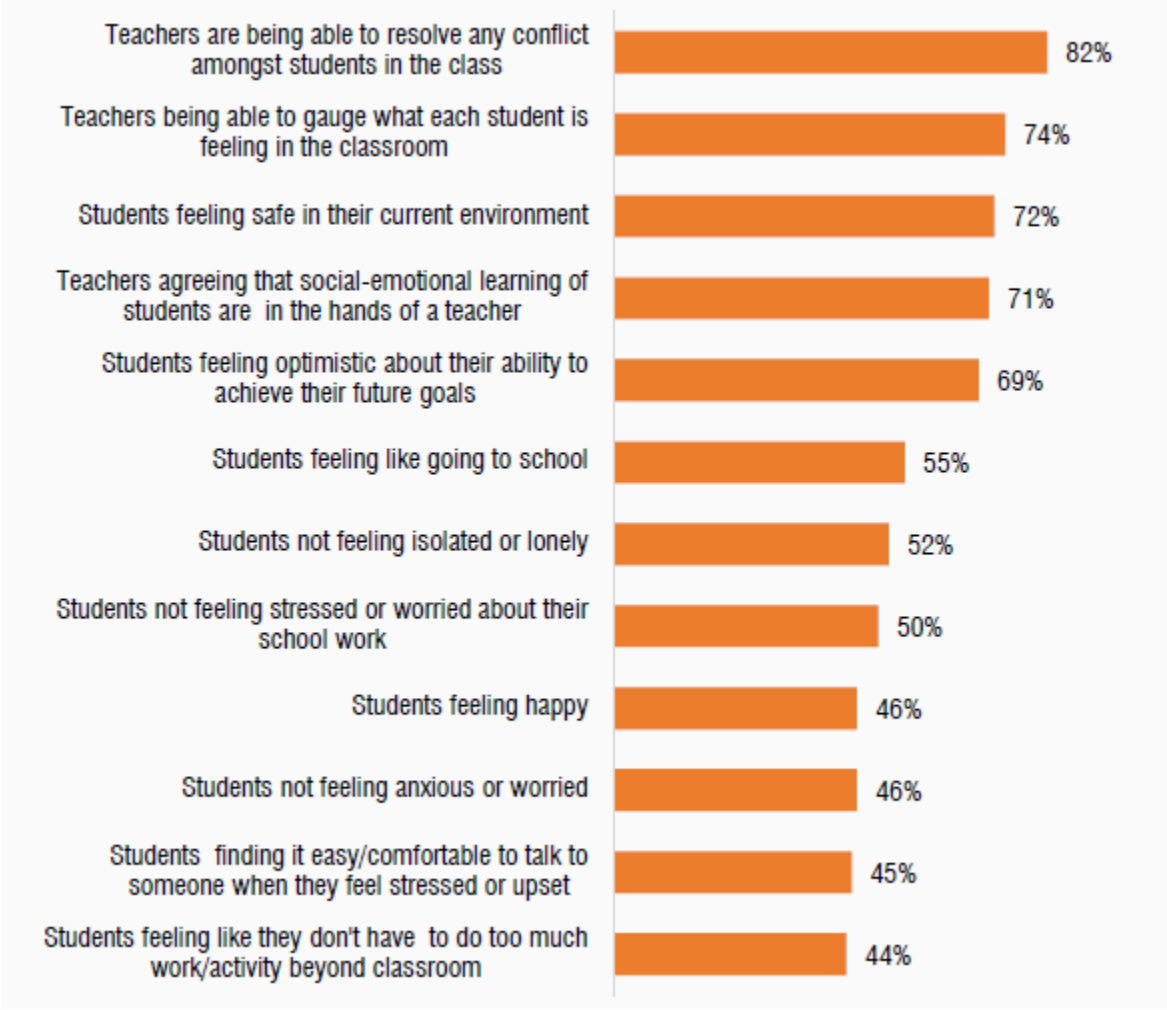
The NEP (2020) criticises the methods of assessment, which often promote rote learning and meaningless practice, contributing to the perception of mathematics as a mechanical subject.

Social Emotional Learning

NEP (2020) recommends focusing on social-emotional learning.

Parakh has provided some data on it based on 2,70,424 teachers and school leaders from 9,74,229 schools.

A bar graph of the teacher's responses is given below:


Teachers are being able to resolve any conflict amongst students in the class
82%
Teachers being able to gauge what each student is feeling in the classroom
74%
Students feeling safe in their current environment
72%
Teachers agreeing that social-emotional learning of students are in the hands of a teacher
71%
Students feeling optimistic about their ability to achieve their future goals
69%
Students feeling like going to school
55%
Students not feeling isolated or lonely
52%
Students not feeling stressed or worried about their school work
50%
Students feeling happy
46%
Students not feeling anxious or worried
46%
Students finding it easy/comfortable to talk to someone when they feel stressed or upset
45%
Students feeling like they don't have to do too much work/activity beyond classroom
44%

As the NEP (2020) has suggested, there is a need to provide additional support to teachers through counselling, teacher training, and school-based

interventions and Manodarpan (https://manodarpan.education.gov.in) to enhance the learning environment.

There is also a need to develop/tackle traits other than self-efficacy and maths anxiety that influence math learning.

Current Challenges in Mathematics Education

The National Curriculum Framework (2023) for school education lists the following challenges for maths education:

- Building a strong foundation in numeracy. This includes a focus on developing number sense, understanding basic operations, and applying them in real-life contexts.
- Rethinking the approach to break down the societal discriminatory beliefs about gender and caste differences.
- Moving away from rote memorisation and towards an interactive and engaging learning experience and focusing on critical thinking and problem-solving.
- Relating maths to students' lives whenever possible.
- Shifting emphasis from symbolic language and formalism in mathematics teaching and learning to the informal argumentation and development of mathematical intuition, which is important for mathematical discovery.
- Learning maths through play and hands-on activities, especially in the Foundational Stage (ages 3-8).
- Tackling maths anxiety.
- Reducing rural-urban, gender, and socioeconomic disparities.

Organisations to Improve Mathematics Education in India

The government of India, states in India, and several NGOs have taken several initiatives to enhance the quality of maths education by designing curriculums, writing textbooks, developing low-cost kits for hands-on activities, tools for assessment, and programmes for professional development of teachers.

Government Organisations

National Council of Educational Research and Training (NCERT)

NCERT is the apex body for advising the Government of India on school education policies, curriculum development, writing textbooks, developing teaching aids and teacher training. It has several constituents: National Institute of Education (NIE), 5 Regional Institutes of Education (RIE), Central Institutes of Educational Technology (CIET), and Pandit Sunderlal Sharma Central Institute of Vocational Education (PSSCIVE), Bhopal, and Parkh. It plays a crucial role in shaping India's education system in line with the National Education Policies.

- It develops textbooks for classes 1-12 and B.Ed.; kits for science and mathematics; learning outcomes and assessment.
- It has designed the national curriculum frameworks (NCF, 2023) for school education and has revised or is revising textbooks to implement the NEP (2020)
- It conducts studies on pedagogy, learning outcomes, and assessment reforms.
- It provides free digital textbooks, audiobooks, and videos in multiple languages through ePathshala.
- It conducts the National Achievement Surveys to evaluate student performance every 4 years.

- It conducts the All India School Education Survey to provide comprehensive data on the state of school education in India on various aspects of educational facilities, enrolment, and teacher qualifications.
- It publishes several journals on school education.
- It designs NIPUN (National Initiative for Proficiency in Reading with Understanding and Numeracy) Bharat guidelines.
- It promotes teaching in the mother tongue/local languages up to class 5.
- It conducts a national talent search and provides scholarships to the selected students for higher studies.
- It collaborates with District Institutes of Education and Training (DIETs) and State Councils of Educational Research and Training (SCERTs), and the National Council of Teacher Education (NCTE).
- It has incorporated **Manodarpan** modules for teacher and student wellbeing in teacher training programmes conducted by NCERT. It has also published a guide for teachers on *Mental Health and Well-Being.* (https://ncert.gov.in/depfe/pdf/Promoting_Mental_Health%20and_Well-being_School.pdf)

Manodarpan: It is an initiative by the Ministry of Education, Government of India, created to provide psychosocial support for the mental health and well-being of students during the COVID outbreak, which continues to be an active and evolving initiative. It is

- Aligned with the NEP (2020) and focuses on mental health and emotional well-being.
- Addresses problems such as stress management and overcoming competitive pressures and exam anxiety for students and stress and burnout for teachers through webinars.
- Its modules are incorporated in teacher training programmes conducted by NCERT.

Central Institute of Educational Technology (CIET)

https://ciet.ncert.gov.in

CIET is a premier institute under the NCERT. Its chief aim is to promote educational technology, especially mass media, singly or in combinations (multimedia packages), to extend educational opportunities and improve the quality of educational processes at the school level. CIET

- Designs and produces media software materials, viz., television/radio (for both broadcast and non-broadcast use) graphics and other programmes for strengthening the transaction of curricular and co-curricular activities at the school level.
- Creates competencies in the development and use of educational software materials through training in areas such as script development, media production, media communication, media research, technical operations, setting up studios, and repair and maintenance of equipment.
- Develops plans for the use of Information and Communication Technologies (ICTs) in education.
- Trains the faculty of Institutes of Advanced Study in Education/Colleges of Teacher Education and District Institutes of Education and Training in the use of educational technology in their teacher education programmes.
- Undertakes research, evaluation and monitoring of the systems, programmes and materials with a view to improving the materials and increasing their effectiveness.
- Documents and disseminates information, materials and media programmes for better utilisation and to function as a clearinghouse/agency in the field of educational technology.
- Advises and coordinates the academic and technical programmes and activities of the State Institutes of Educational Technology (SIETs) set up by the Ministry of Education in states of India.
- Produces e-learning resources, including videos, audio programmes, and interactive multimedia for schools.
- Manages platforms like Digital Infrastructure for Knowledge Sharing (DIKSHA) and e-Pathshala.

- Coordinates with Doordarshan (DD) and All India Radio (AIR) to broadcast educational programs under Gyan Darshan (TV channel for education) and Gyan Van (educational radio programmes).
- Conducts workshops and online courses for teachers on digital pedagogy, cyber safety, and e-content creation.
- Collaborates with the National Initiative for School Heads and Teachers' Holistic Advancement (NISHTHA) for capacity building.
- Studies the impact of technology in education.
- Promotes Artificial Intelligence (AI), Augmented Reality (AR), and Virtual Reality (VR) in learning.
- Develops digital materials for the National Institute of Open Schooling (NIOS) and other Open Distance and Learning (ODL) programmes.
- Assesses schemes like PMeVIDYA and the Digital India Campaign in schools.
- Hosts the National Conference on ICT in School Education annually.
- Collaborates with SCERTs, Kendriye, Navoday, and other educational bodies.

Parakh

It is an initiative by the Indian government aimed at standardising school education assessments across the country.

Key objectives of Parakh are to

- Establish consistent evaluation frameworks for schools to ensure fairness and comparability.
- Move away from rote learning to focus on critical thinking, problem-solving, and application of knowledge.

- Introduce a comprehensive student evaluation system that tracks academic as well as co-curricular progress.
- Equip educators with tools and methodologies for effective assessments.
- Will now conduct periodic national-level assessments (National Achievement Survey) to monitor learning outcomes.

National Council of Teacher Education (NCTE)

NCTE

- Recognises teacher education institutions and programmes (e.g. B.Ed, D.El.Ed, M.Ed).
- Sets norms and standards for teacher training courses.
- Recommends frameworks for teacher education programmes.
- Ensures alignment with national education policies.
- Monitors teacher training institutions to maintain standards.
- Strengthens teacher training through digital platforms (DIKSHA, SWAYAM (Study Webs of Active Learning for Young Aspiring Minds)).
- Promotes equitable access to teacher education.
- Takes action against substandard or unauthorised institutions.
- Works with central/state governments to improve teacher education.
- Promotes innovations like integrated teacher education programmes.
- Supports research in teacher education.
- Collaborates with organisations like NCERT and the National Institute of Educational Planning and Administration (NIEPA).

It faces the challenges of ensuring uniform quality across private and government institutions, addressing teacher shortages in rural areas, and adapting to technological advancements in education.

State Councils of Educational Research and Training (SCERTs)

The SCERTs

- Adapt the national curriculum framework (NCF, 2023) provided by the NCERT to meet state-specific needs, incorporating local languages, culture, and history.
- Develop state-specific textbooks and educational materials aligned with regional contexts.
- Organise training sessions for teachers to update pedagogical skills and implement new teaching methodologies.
- Oversee District Institutes of Education and Training (DIETs) to ensure effective grassroots-level teacher education.
- Conduct studies to evaluate educational policies, teaching strategies, and student learning outcomes.
- Partner with schools to test new educational programs or tools, ensuring evidence-based improvements.
- Design and implement evaluation systems to assess student performance and inform targeted interventions.
- Track educational standards to identify gaps and recommend improvements.
- Develop e-learning modules and tools suited to the state's infrastructure, such as offline-friendly content or regional-language platforms.
- Work with NCERT, SCERTs, and NGOs to align goals and share best practices.

- Promote education in regional languages to enhance accessibility for diverse student populations.
- Address disparities through programmes for marginalised communities, girls' education, and children with disabilities.

Padhe Bharat Badhe Bharat (PBBB)

PBBB is a subprogram under the Samagra Shiksha Abhiyan launched by the Ministry of Education, Government of India, in 2014. It is now under Samagra Shiksha. NCERT & SCERTs provide support in material development and teacher training. Its aim for classes 1-3 is to

- Enhance age-appropriate reading, writing, and comprehension abilities.
- Strengthen foundational numeracy skills through hands-on activities, games, and real-life applications to make maths engaging.
- Foster a habit of reading and critical thinking from an early age through joyful learning through libraries, reading corners, and storytelling.
- Equip teachers with effective pedagogical techniques for foundational learning.
- Involve parents and communities to support children's learning at home.

However, it faces challenges of uneven implementation across states, improvement of teacher training and monitoring, and ensuring learning continuity in rural and remote areas.

Rashtriya Avishkar Abhiyan (RAA): It is a flagship initiative launched by the Ministry of Education, Government of India, in 2015 to foster scientific temper, innovation, and curiosity among school students (classes 6–12). Aligned with the National Education Policy (NEP) 2020, RAA aims to bridge the gap between formal education and hands-on learning in science, technology, engineering, and mathematics (STEM) fields.

Objectives of RAA

Its Objectives are to

- Encourage research-mindedness through experiments, projects, and collaborations.
- Reduce fear of STEM subjects by making them engaging and relatable.
- Strengthen teacher capacity for innovative pedagogy.
- Link classrooms to real-world scientific applications.

Means for achieving objectives

- Establishment of science/mathematics clubs and innovation hubs to promote inquiry-based learning in science and mathematics with the help of Indian Institutes of Technology (IITs), Indian Institutes of Science Education and Research (IISERs), and NGOs for tools and labs.
- Organisation of national-level events (e.g. National Children's Science Congress).
- Organisation of workshops on activity-based teaching (e.g. low-cost experiments).
- Use of STEM kits in teaching.
- Conduct campaigns on STEM careers for parental awareness.
- Collaboration with local industries and science centres (e.g. Nehru Planetarium).
- Implementation with the help of NCERT (for curriculum support), Kendriya/Jawahar Navodaya Vidyalayas, State SCERTs (for local adaptation), and DIKSHA and SWAYAM Prabha (for resource sharing).

Impact & achievements of RAA

- It engages more than 10 million students annually.

- It has established more than 500 hubs (e.g. Atal Tinkering Labs).
- It has improved participation in competitions (e.g. National Talent Search Examination, Indian National Olympiad).
- It has enhanced gender inclusivity in STEM (40% female participation in 2023).

However, it faces the challenges of uneven implementation across states, resource gaps in rural schools (lack of labs/trained teachers), and limited industry linkages in remote areas.

National Initiative for Proficiency in Reading with Understanding and Numeracy (NIPUN)

It is a government programme in India focused on ensuring foundational literacy and numeracy skills for all children by the end of Grade 3. Launched in 2021, it aims to achieve this by 2026-27. It is to be implemented by a 5-tier mechanism (National-State-District-Block/Cluster-School) to ensure cross-linkages and effective implementation. It aims to have

- Clear, grade-wise targets for literacy (e.g., decoding texts, comprehension) and numeracy (e.g., number sense, arithmetic operations).
- Empower teachers through FLN-focused pedagogies (e.g., phonics, play-based math activities) and training programs for the use of the DIKSHA Platform for digital resources and continuous professional development.
- Redesign textbooks and workbooks emphasising interactive, child-centric methods, e.g., integration of stories, games, art, and local contexts to make learning engaging.
- Conduct regular, competency-based evaluations to track individual progress and the National Achievement Survey (NAS) to monitor systemic outcomes.

- Conduct awareness campaigns to emphasise the importance of foundational skills through School Management Committees (SMCs) and volunteers to support at-home learning.
- Use apps like e-Pathshala and interactive content for remedial support and Balvatika to prepare children for Grade 1.
- Collaborate with NGOs (e.g., Pratham, Akshara Foundation) and international agencies like UNICEF).
- Coordinated efforts by states/UTs to adapt NIPUN guidelines to local contexts.
- Open resource centres in schools for teacher collaboration and capacity-building (NIPUN Bharat Kendras)

It has trained over 5 million teachers in FLN strategies as of 2023, and Parakh has conducted a baseline survey (FLN, 2022: https://ncert.nic.in/fls.php) to find the present status of foundational literacy and numeracy, variations across states and regions, gender and social disparities, attendance and school readiness, and teacher preparedness.

Swayam Prabha

It is a flagship initiative of the Government of India under the Ministry of Education, designed to provide free, high-quality educational content to learners across the country, particularly in remote and underserved areas. Launched in 2016 as part of the Digital India and NEP (2020) goals, it uses Direct-to-Home (DTH) television channels to broadcast curriculum-aligned lessons, reducing reliance on internet connectivity. It has 32 free DTH channels dedicated to specific educational levels (e.g. school education, higher education, vocational training) and subjects, including maths, science, and STEM. Content is broadcast 24/7, with lessons repeated multiple times for flexibility. It reaches over 10 million households daily, including tribal and rural communities.

National Institute of Open Schooling (NIOS)

NIOS provides flexible learning options for students who cannot attend regular schools, including maths courses.

Open Education Resources (OER)

OER are freely accessible, openly licensed teaching and learning materials that support equitable education. Aligned with the National Education Policy (NEP) 2020 and Digital India, OER initiatives aim to bridge gaps in access, affordability, and quality.

Several NGOs and private institutions have contributed to OER.

These are provided through

- DIKSHA (https://diksha.gov.in): It provides free e-textbooks, teacher training modules, and interactive content (classes 1-12) and supports multiple Indian languages. These are aligned with NCF 2023 and state curricula.
- e-Pathshala (https://epathshala.nic.in): It provides NCERT textbooks, audiobooks, videos, and flipbooks. These are available in English, Hindi, and regional languages.
- SWAYAM (Study Webs of Active Learning for Young Aspiring Minds) (https://swayam.gov.in): it provides Free online courses for school to university levels. It was developed by All India Council for Teacher Education (AICTE), NCERT, and top universities
- National Digital Library (NDL) (https://ndl.gov.in): It has over 6.5 million resources (books, lectures, simulations) to support lifelong learning.
- The National Repository Of Open Educational Resources (NROER) (https://nroer.gov.in): It features interactive modules, videos, and lesson plans and focuses on STEM and multilingual education.
- PhET India Simulations: It provides interactive maths/science simulations and is hosted on DIKSHA.

- TicTacLearn (http://ticktaclearn): It has free animated video lessons in more than 10 regional languages.
- Khan Academy (India): It provides localised content in Hindi and regional languages.
- Pratham Books' StoryWeaver: It provides open-source multilingual storybooks for foundational literacy.

However, the use of OER involves several challenges, viz., limited internet access in rural areas, varying standards across OER platforms, teachers'/students' unawareness of OER availability, and paucity of content in tribal and lesser-spoken languages.

Impact of Government Initiatives

The percentages of students at different performance levels in the National Achievement Survey in 2017 and 2021 for classes 3, 5, and 8 are given in Table 1.5. In 2017, the students in grade 10 were not a part of the survey.

There is a considerable drop-out in the percentage of students in proficient and advanced categories from 52 to 42 in grade III, 44 to 25 in grade 5, and 40 to 27 in grade 8. However, there was Covid in between.

The PARAKH Rashtriya Survekshan Survey (2024) reported the average scores of students in classes 3, 6, and 9 as 60%, 46%, and 37%, respectively.

It is difficult to compare the performance of students in this survey with that of students in NAS (2017 and 2021), as tests, classes, and reporting of data are different. But low mastery of foundational skills by Class 3, along with the significant decline in learning outcomes from Class 3 to 6 and again from Class 6 to 9, highlights the urgent need to strengthen learning across all grade levels.

Table 1.5: The percentage of students in different categories of performance for classes 3, 5, and 8 (NAS 2017 & 2021)

Class	Year	Below Basic	Basic	Proficient	Advanced
3	2017	13	34	36	16
	2021	23	34	30	12
5	2017	18	39	31	13
	2021	34	41	19	6
8	2017	18	43	29	11
	2021	27	46	19	8

Non-Governmental Organisations

Pratham: It is one of India's largest and most influential NGOs focused on improving education for underprivileged children. Founded in 1994 in Mumbai, Pratham operates across 23 states and is renowned for its low-cost, scalable interventions to enhance learning outcomes, especially in foundational literacy and numeracy (FLN). It

- Organises learning camps that provide intensive remedial sessions for lagging students using activity-based learning and peer-group teaching.
- Trains caregivers to support early learning at home.
- Offers vocational training (e.g. healthcare, retail, hospitality) for youth (age group 15–25).
- It has apps like PraDigi for tablet-based learning in regional languages.
- Conducts Annual Status of Education Reports (ASER)—India's largest citizen-led household survey on children's learning levels.

- Influences national policies, e.g., NEP 2020's focus on FLN.
- Its flagship programme is teaching maths and reading skills at the right level (TaRL), which groups children by learning level (not class) for targeted instruction. This programme has been adopted by governments in UP, Bihar, and Gujarat.
- Helps out-of-school girls and women to complete secondary education.
- Has established community libraries to foster a reading culture.
- Uses WhatsApp, IVR calls, and community volunteers for remote education.
- Works with state education departments to scale programmes.

It directly impacts more than 10 million children annually, collaborates with governments to scale programmes nationwide, and has been recognised with the WISE Prize for Education Innovation (2012) and Libraries of Hope Award (2020).

However, it faces the challenges of sustaining quality while expanding, high turnover of community volunteers, teachers' retention, and very limited tech access in rural areas. It also needs to expand TaRL to more states, strengthen STEM education for girls, and integrate AI-based adaptive learning tools.

- **Agastya Learning Portal**

Agastya International Foundation is a Bengaluru-based nonprofit organisation dedicated to transforming STEM (science, technology, engineering, and mathematics) education in India through experiential, hands-on learning. Founded in 1999, it focuses on sparking curiosity, creativity, and critical thinking, particularly in underserved rural and urban schools. It

- Provides high-quality, interactive maths and science education to economically disadvantaged students.

- Trains teachers to adopt innovative, student-centred pedagogies.
- Promotes experiential learning, multidisciplinary thinking, and 21st-century skills.
- Provides vans equipped with portable maths and science models that travel to rural schools.
- Provides mobile labs that have geometry, algebra, and measurement kits.
- Conducts teacher training workshops on how to design lessons using low-cost manipulatives.
- Uses strategies to reduce maths anxiety and encourage experimentation, and links maths to real-world problems (e.g. budgeting, data analysis).
- Uses peer-learning models where older students (Grade 8+) teach younger ones under teacher supervision by training student leaders to explain concepts like fractions or angles using Agastya's kits.
- Provides online modules and videos for remote learning (e.g., geometric constructions, probability).
- Has a YouTube channel that provides free tutorials on activity-based maths teaching.
- Has Agastya's networks that connect educators.
- Provides mobile labs to bridge infrastructure shortages in rural schools.

It has reached more than 13 million students across 22 Indian states, improved maths scores by 25% in rural schools using Agastya's models in Andhra Pradesh, and led to higher student participation in algebra after hands-on workshops in Karnataka, according to teachers' reports.

- **Tata Institute of Fundamental Research (TIFR)**

  TIFR is India's premier research institution for advancing fundamental science and mathematics. Its Homi Bhabha Centre for Science Education (HBCSE) is dedicated to improving STEM education in India by emphasising understanding, equity, and bridging the gap between research and classroom practice. HBCSE

  - Identifies and trains students for the International Mathematical Olympiad (IMO).
  - Develops problem-solving resources for use by teachers nationwide.
  - Design activity-based learning modules aligned with NEP (2020).
  - Publishes books.
  - Provides workshops on inquiry-based pedagogy and fostering mathematical thinking.
  - Collaborates with NCERT and state boards to reform maths curricula.
  - Investigates how students learn math concepts (e.g. spatial reasoning, algebra).
  - Informs national education policies through evidence-based recommendations.
  - Promotes collaborative problem-solving sessions for students and teachers.
  - Provides open access to lectures and papers for educators (e.g. TIFR's online archives).
  - Organises public lectures by renowned mathematicians.

- **The Azim Premji Foundation**

It was established in 2001 by Azim Premji. It is one of India's largest non-profit education initiatives. It focuses on improving equity and quality in public education through systemic reforms, teacher capacity building, and community engagement. It works with the state governments of Karnataka, Uttarakhand, Rajasthan, Chhattisgarh, Telangana, and Madhya Pradesh:

- To improve Foundational literacy & numeracy (FLN)
- Track students' progress in foundational skills (reading, maths).
- Develops curricula.
- Provides teacher training and professional development for in-service teachers.
- Conducts workshops on activity-based learning, assessment reforms, and inclusive education.
- Trains for school leadership and governance.
- Provides online & blended learning resources via DIKSHA, MOOCs, and Azim Premji Foundation (APF) portals.
- Strengthens District Institutes of Elementary Teacher Training (DIETs)
- Conducts large-scale studies on learning outcomes, teacher effectiveness, and education equity.
- Influences national education policies and national curriculum frameworks.
- Develops open-source digital tools for teachers and students.
- Promotes blended learning in rural and tribal areas.
- Works with parents, local governments, and NGOs to improve school participation.

- Focuses on marginalised groups, e.g. girls, tribal communities, children with disabilities.
- Focuses on research-driven solutions for India's education challenges.
- Provides open educational resources (OERs) such as textbooks, lesson plans, assessments for teachers, and research papers, which are hosted on DIKSHA and APF portals.

It has trained more than 700,000 teachers in innovative pedagogies, improved learning outcomes in more than 5,000 government schools, and published more than 100 research papers on education equity and pedagogy.

- **Ganitha Kalika Andolana (GKA)**

  It is a large-scale programme for improving maths education through collaboration between the NGO and the government. It is designed to improve numeracy outcomes for primary students in India by using hands-on, activity-based learning with manipulatives provided in maths kits. Teachers are trained to use these kits and incorporate group work and discussions.

  It has reached more than 4.5 million students across more than 46,000 schools, improving numeracy skills by 25–30% (external evaluations). It has been featured in the Numeracy at Scale study, which focuses on identifying the characteristics of 8 successful programmes from around the world in low- and middle-income contexts that have improved early grade numeracy outcomes at scale.

- **Nanhi Kali**

  Nanhi Kali was initiated by the K. C. Mahindra Education Trust in partnership with the Nandi Foundation. It is one of India's most prominent girl-child education programmes. Since its launch in 1996, it has empowered underprivileged girls through academic support, mentorship,

and holistic development to ensure they complete 10 years of schooling. It focuses on

- Academic support by providing daily after-school tutoring for maths, science, and languages.
- Providing tablets with interactive content by Mindspark.
- Supplying school books, uniforms, and shoes.
- Using female role models to guide girls on career choices, health, and confidence-building.
- Organisation of workshops on topics like hygiene, financial literacy, and STEM awareness.
- Educates families on the importance of girls' education.
- Offers scholarships to prevent dropouts.

It has supported more than 550,000 girls from low-income families and high dropout-risk communities from classes 1-10, who are selected from government schools of 12 states. It has a 92% school retention rate (vs. the national average of 86%), with 65% achieving grade-level competency in core subjects.

It has been featured in the National Institution for Transforming India (NITI) Aayog's Aspirational Districts Programme and the Numeracy at Scale study, which focuses on identifying the characteristics of 8 successful programmes from around the world in low- and middle-income contexts that have improved early grade literacy and numeracy outcomes at scale. You can also watch the video for key takeaways of this study on ▶ What Works to Improve Numeracy at Scale? Key Takeaways from 6 …

However, it faces many challenges, e.g. early marriage, household duties, gender biases, lack of schools in rural areas, and limited internet access.

- **EduBridge** (edubridgeindia.com)

It is a social enterprise committed to enhancing the quality of education in rural India through innovative teacher training programmes. It is aligned with NEP (2020) and NCF (2023). EduBridge focuses on

- Equipping teachers in rural areas with pedagogical skills, technological tools, and community-centric strategies to address the unique challenges of underserved regions to bridge the urban-rural education gap by empowering teachers with 21st-century teaching methodologies.
- Conducting blended learning workshops.
- Combining in-person workshops with digital modules, which are accessible via low-bandwidth platforms.
- Covering foundational numeracy in maths.
- Using activity-based strategies for teaching maths concepts (e.g. fractions, place value).
- Providing training in regional languages (e.g. Marathi, Odia, Telugu) to improve student comprehension.
- Using low-cost tech tools (e.g. smartphones, projectors) for interactive lessons.
- Collaborating with NGOs, self-help groups, and government schools to tailor programmes to regional needs.
- Training teachers to engage parents in supporting home learning (e.g. maths games using household items).
- Using mobile teams to conduct on-site training in remote areas, addressing infrastructure and connectivity gaps.
- Distributing low-cost teaching aids (e.g. flash cards, fraction tiles) made from local materials.

- Providing an offline access platform with video tutorials, lesson plans, and assessment tools through the EduBridge App.
- Sharing daily teaching tips, activity ideas, and NEP/NCF updates with rural teachers using WhatsApp Groups
- Shifting from rote memorisation to problem-solving and critical thinking.
- Addressing gender, caste, and disability barriers through sensitisation workshops.
- Linking maths to real-life skills (e.g. budgeting, measuring agricultural land).
- Partnering with state governments (e.g. Rajasthan, Chhattisgarh) to scale the programmes under Samagra Shiksha Abhiyan (SSA).

It has trained more than 1,200 teachers in activity-based maths instruction, leading to a 35% improvement in student numeracy scores in Maharashtra, and its mobile camps equipped more than 500 schools with maths kits, reducing dropout rates in classes 6–8 in Odisha.

It has tried to meet the challenges of internet access, language diversity, and teacher retention with offline apps and Straight to Your Doorstep Card-Based Content Delivery, training in more than 10 regional languages, and incentives like certificates and community recognition.

- **Jodo Gyan** (https://youtube.com/jodogyan)

It is a Delhi-based NGO dedicated to transforming maths education through activity-based learning, low-cost teaching tools, and teacher training. Founded in 1999, it aligns with India's NEP 2020 and NCF 2023 goals of fostering conceptual understanding, critical thinking, and joyful learning.

Jodo Gyan kits has improved foundational numeracy in more than 500 government schools in Delhi and has led to a 30% increase in student participation in class after using fraction kits in rural Rajasthan

Key Takeaways

1. ASER data suggests the need to provide for mastery of foundational skills necessary for daily needs and further learning as recognised by NEP (2020).

   There are several government and nongovernment organisations working on it; there is a need to check the implementation and effectiveness of their programmes.

2. The National Achievement Survey and Parakh's Rashtriya Suevekshan data highlight the steep decline in higher grades and thus the urgent need to strengthen learning across all grade levels. As maths is a hierarchical subject.

3. CTET data again suggests the need to strengthen mathematics education and teacher training programmes to increase the number of candidates who can qualify in CTET and TETs to address the shortage of teachers and improve the quality of teachers.

4. The data on teaching practices suggests that it is crucial to expand the use of practices enumerated there and integrate more research-based approaches that have proven effective in enhancing mathematics education.

5. NGOs which work in some states have incorporated many strategies other than providing kits for students and organising workshops for teachers, which are also undertaken by government institutions for improving math education that can be emulated or expanded to other states. For example, for

- **FLN:** Organising learning camps for lagging students using activity-based learning and teaching at the right level by grouping students by their level and not age by Pratham.
- **Learning from peers or older students:** implemented by Pratham and Agastya Learning Portal.
- **Parent and community involvement for learning outside school:** implemented by Pratham, Nanhi Kali, and the Azim Premji Foundation.
- **Development of resources for higher-order thinking:** as done by the Azim Premji Foundation.
- **Equity:** Improving girls' participation by Nanhi Kali; providing for blended learning by the Azim Premji Foundation for rural and tribal areas; providing vans equipped with portable maths and science labs that travel to rural schools by the Agastya Learning Portal.
- **Networking for educators** by Agastya Learning Portal.
- **Research** on effective learning strategies, teacher effectiveness, and equity by the Tata Institute of Fundamental Research.

2

Hattie's Research on Variables That Impact Achievement

In a groundbreaking study, John Hattie (2023) has provided a database showing how more than 350 variables may influence achievement. It is based on more than 2100 meta-analyses from 130,000 studies involving more than 400 million students. The variables relate to the domains of students, home, school, classroom, teacher, curriculum, student learning, teaching strategies, technology, school, and out-of-school variables. For each of these, he provides the number of meta-analyses, the number of studies, and the estimated total number of students on which it is based; the weighted effect size; the standard error; the robustness of the effect size; and a summary of these by a visual thermometer/temperature gauge, which also gives the range in effect sizes.

The effect size tells us how *big* the difference is between the two groups who followed different approaches to teaching or learning so that we can judge the extent of the effectiveness of the approach. It is denoted by Cohen's d, which is defined by

$$\text{Cohen's d} = \frac{Mean\ of\ the\ experimental\ group - Mean\ of\ the\ control\ group}{Standard\ deviation\ of\ pooled\ or\ control\ group}$$

If sample sizes are small, Hedges' g (a bias-corrected version of Cohen's d) is used.

It is given by

$g = d \times (1 - \frac{3}{4(n_1+n_2)-9})$, where $n_1\ and\ n_2$ are the sample sizes of the experimental and control groups, respectively.

For correlational studies, Pearson's r between the two variables is calculated. If results from correlational and experimental studies are to be combined for finding the overall effect size. Then r can be converted into effect size d by

$$d = \frac{2r}{\sqrt{1-r^2}}$$

For more on effect size, see Appendix 2.

Meta-analysis: objectively examines published data from many studies on the same research topic identified through a literature search. It can reveal patterns hidden in individual studies through the use of rigorous statistical methods and can yield conclusions that have a high degree of reliability. It is a method of analysis that is especially useful for gaining an understanding of complex phenomena when independent studies have produced conflicting findings.

Statistical methods are used to find the weighted average effect size of multiple independent studies on the same topic to draw an overall conclusion, giving more weight to studies with larger sample sizes.

Interpretation of average weighted effect size: The average weighted effect size for all the variables in Hattie's study was 0.40. Hattie suggested using an effect size of 0.40 as a **hinge point** to judge the effectiveness of treatment or the impact of the variable. He also advised not to get wedded to the hinge point of 0.40, as sometimes smaller effect sizes may also be valuable, depending on context. Use local knowledge as to what is best for the context.

Robustness of meta-analysis refers to the stability of the average effect size when some studies are included or excluded from it. It is on a scale of 1-5.

We will restrict ourselves to the relationship between variables related to teachers, students, and classroom climate on achievement.

We present the data from Hattie's study next. We will consider only variables whose effect size is equal to or greater than 0.40 and has a robustness of 3 or more. For the effectiveness of other variables, see Hattie, J. (2023) or www.visiblelearningmetax.com/Influences

For a description of variables, see glossary in www.visiblelearning.com

Studies on Teacher Characteristics and Teaching Strategies

Hattie's research shows that teachers are the most significant factor in student achievement.

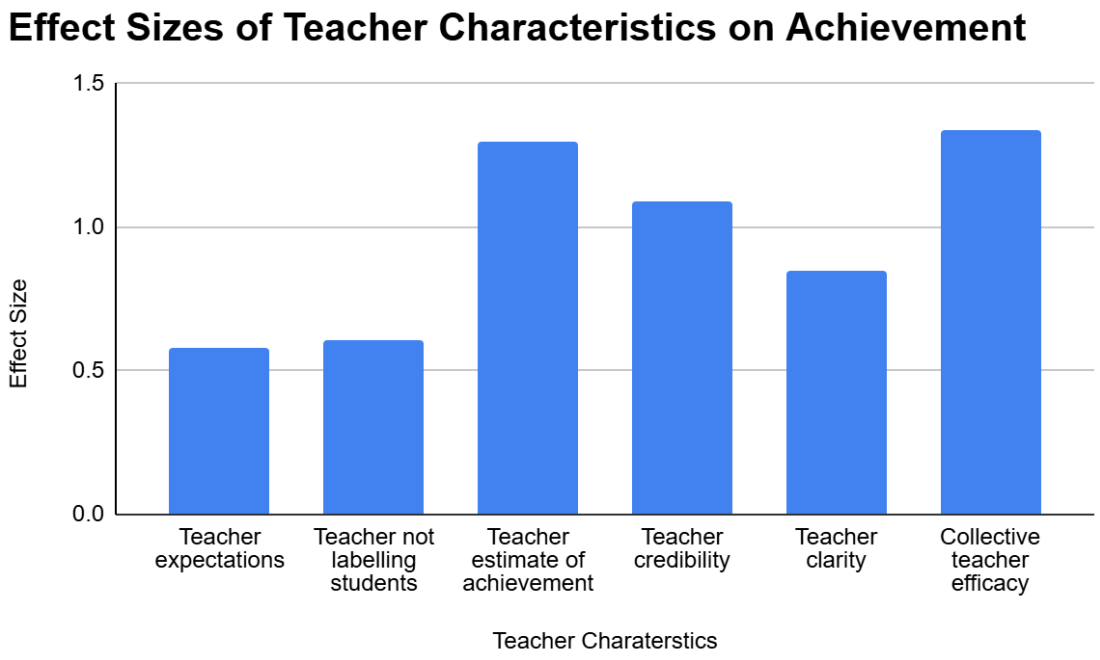
Teachers' characteristics

The most influential characteristics of teachers that have a high impact on student achievement are given in Table 2.1

Table 2.1: Effect sizes of teacher's characteristics on achievement

Characteristic	Weighted effect size	Robustness
Teacher expectations	0.58	4
Teacher not labelling students	0.61	2
Teacher estimate of achievement	1.30	3
Teacher credibility	1.09	2
Teacher clarity	0.85	3
Collective teacher efficacy	1.34	2

A bar graph of effect sizes of teacher characteristics is given below:



Teaching Intent

Students' achievement depends on how well the teacher aligns the curriculum content, learning intentions, success criteria, assessment, and feedback in his teaching.

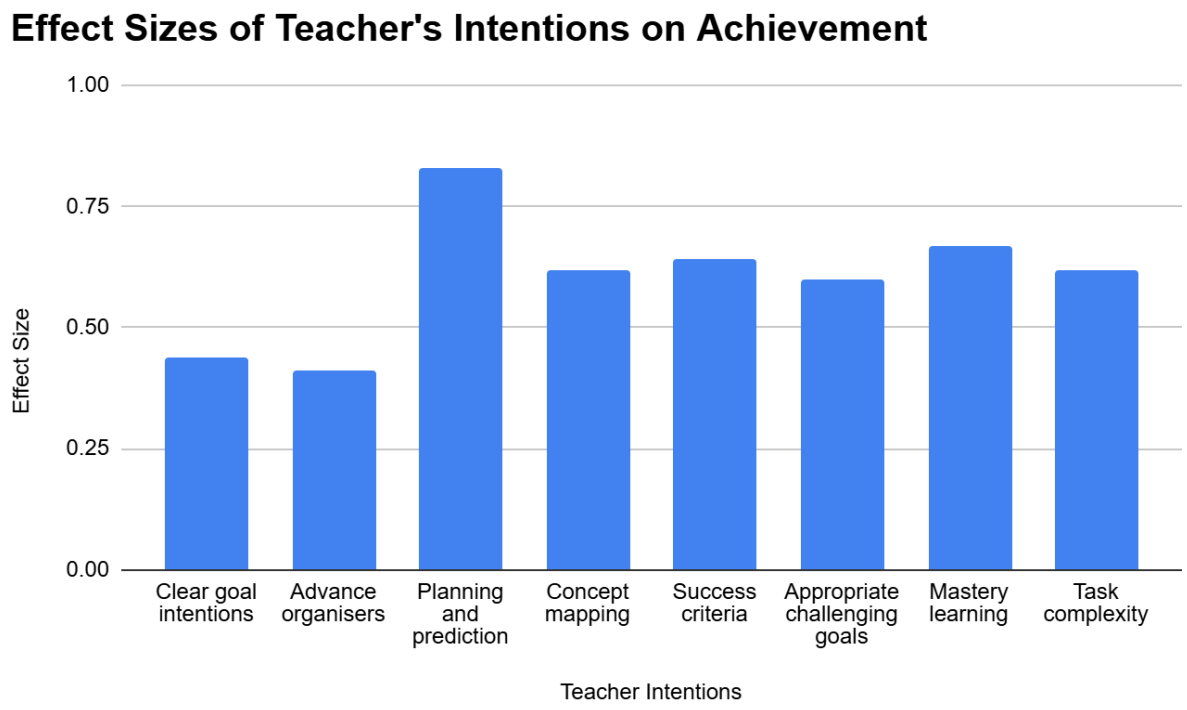
The variables with a high impact are given in Table 2.2.

Table 2.2: Effect sizes of teachers' intent on achievement.

Strategy	Weighted effect size	Robustness
Clear goal intentions	0.44	4
Advance organisers	0.41	4
Planning and prediction	0.83	3
Concept mapping	0.62	5

Success criteria	0.64	2
Appropriate challenging goals	0.60	4
Mastery learning	0.67	4
Task complexity	0.62	4

A bar graph of effect sizes of teachers' intentions is given below:



The effectiveness of teaching strategies for surface knowledge and deep knowledge

Teachers use different strategies for teaching. The strategies with a high impact for surface knowledge and deep knowledge are given in Tables 2.3 and 2.4.

Table 2.3: Effect sizes of teaching strategies for surface knowledge on achievement

Strategy	Weighted effect size	Robustness
Direct instruction	0.56	5
Explicit teaching	0.63	5
Response to intervention	0.73	4
Use of Adjunct aids	0.64	4

A bar graph of effect sizes of teaching strategies for surface knowledge is given below:

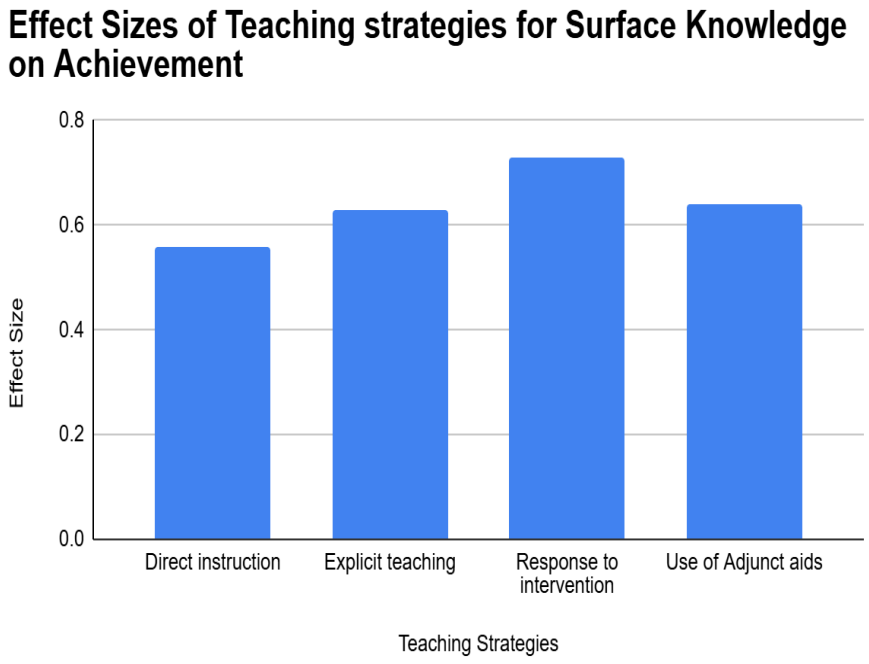
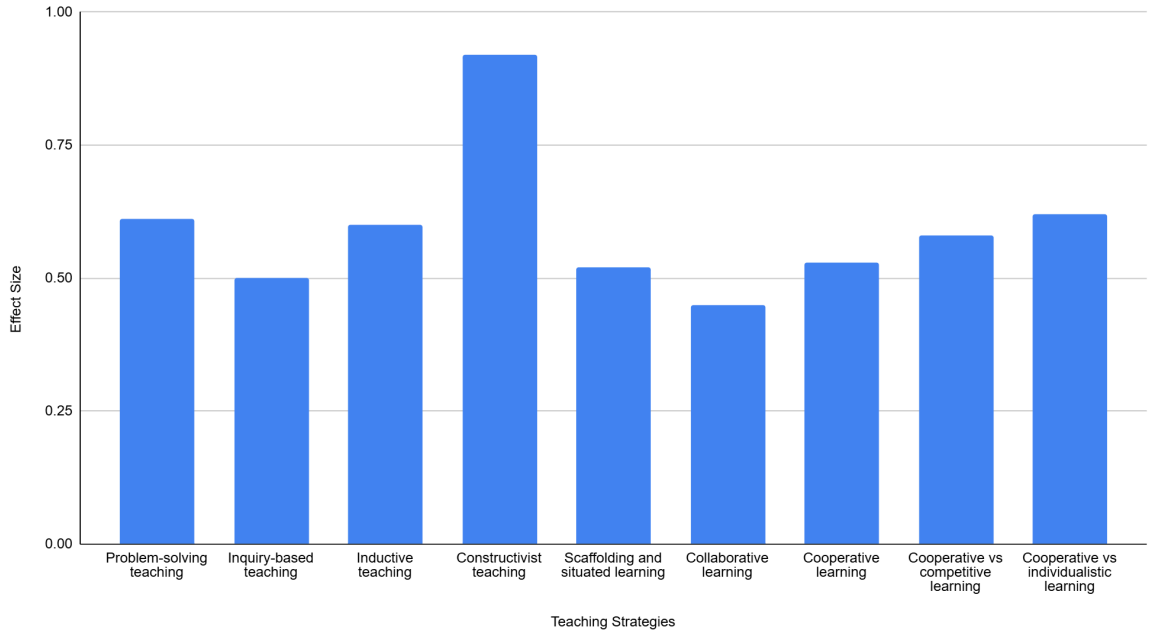


Table 2.4: Effect sizes of different teaching strategies for deep understanding on achievement

Strategy	Weighted effect size	Robustness
Problem-based learning	0.45	5
Problem-solving teaching	0.61	5
Inquiry-based teaching	0.50	4
Inductive teaching	0.60	3
Constructivist teaching	0.92	3
Scaffolding and situated learning	0.52	4
Collaborative learning	0.45	4
Cooperative learning	0.53	5
Cooperative vs competitive learning	0.58	4
Cooperative vs individualistic learning	0.62	4

A bar graph of effect sizes of teaching strategies for deep understanding is given below:

Effect Sizes of teaching strategies for deep understanding



Studies on Students' Characteristics in the Cognitive and Affective Domain and Learning Strategies

The achievement of students also depends on what abilities they had or have acquired in the cognitive domain, their learning strategies, and traits in the affective domain that affect learning.

The variables having a high impact in the cognitive domain are given in Table 2.5, those in the affective domain in Table 2.6, and learning strategies in Table 2.7.

Table 2.5: Effect sizes of students' characteristics in the cognitive domain for achievement

Characteristic	Weighted effect size	Robustness
Prior ability and intelligence	0.96	5
Prior achievement	0.73	5
Working memory strength	0.63	4
Creativity	0.40	4
Critical thinking	0.84	3

A bar graph of effect sizes of students' characteristics in the cognitive domain for achievement is given below:

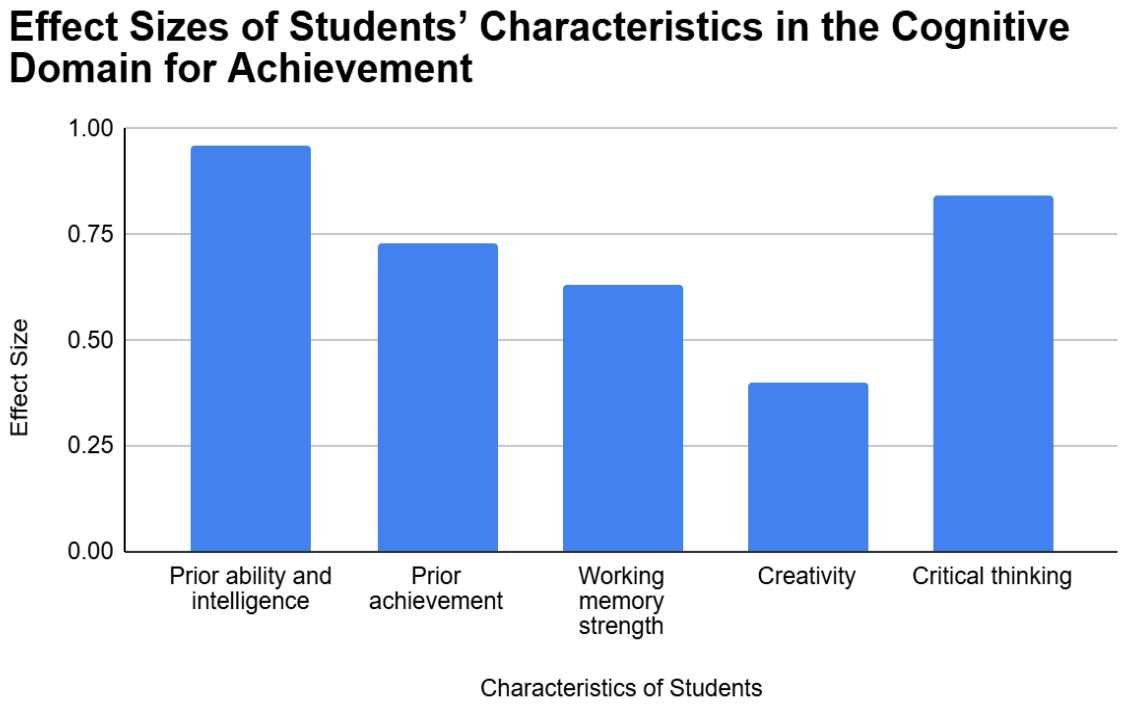


Table 2.6: Effect sizes of the achievement for students' characteristics in the affective domain

Characteristic	Weighted effect size	Robustness
Growth mindset	0.19	4
Persistence	0.41	5
Self-efficacy	0.64	5
Curiosity	0.74	2
Anxiety	-0.40	5
Procrastination	-0.41	3
Positive self-concept	0.51	5
Self-control	0.66	3
Emotional intelligence	0.50	5
Emotions	0.61	2
Enjoyment	0.50	3
Anger	-0.65	2
Boredom	-0.46	3

A bar graph of the effect sizes of the achievement for students' characteristics in the affective domain is given below:

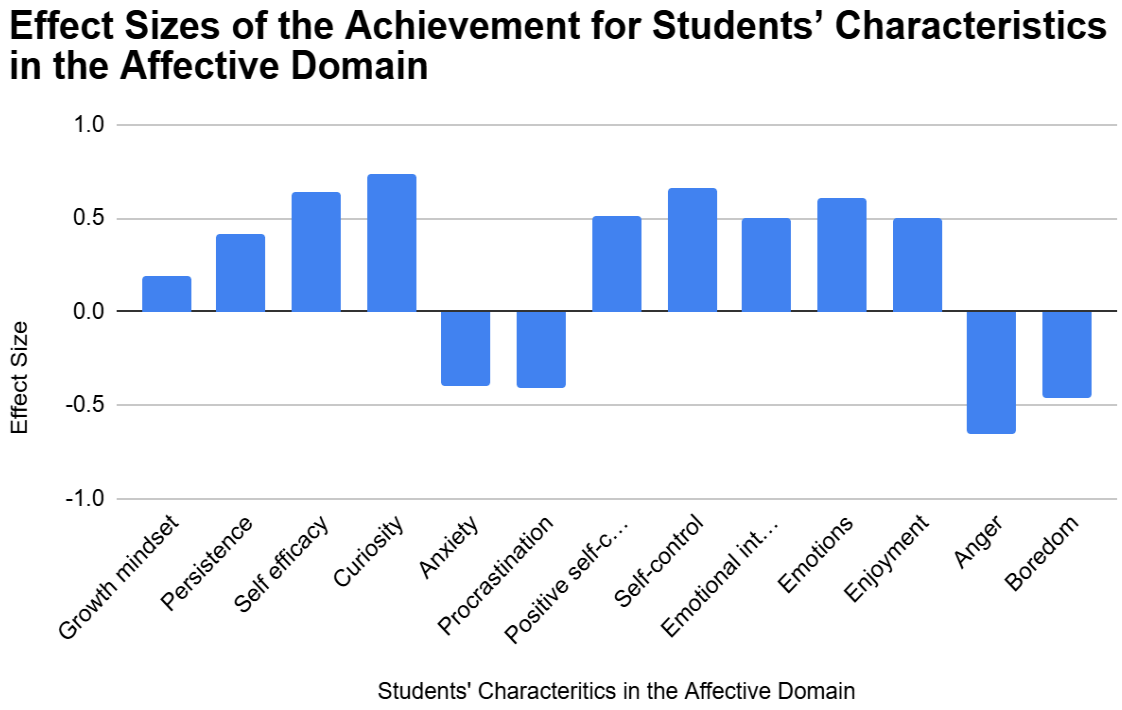
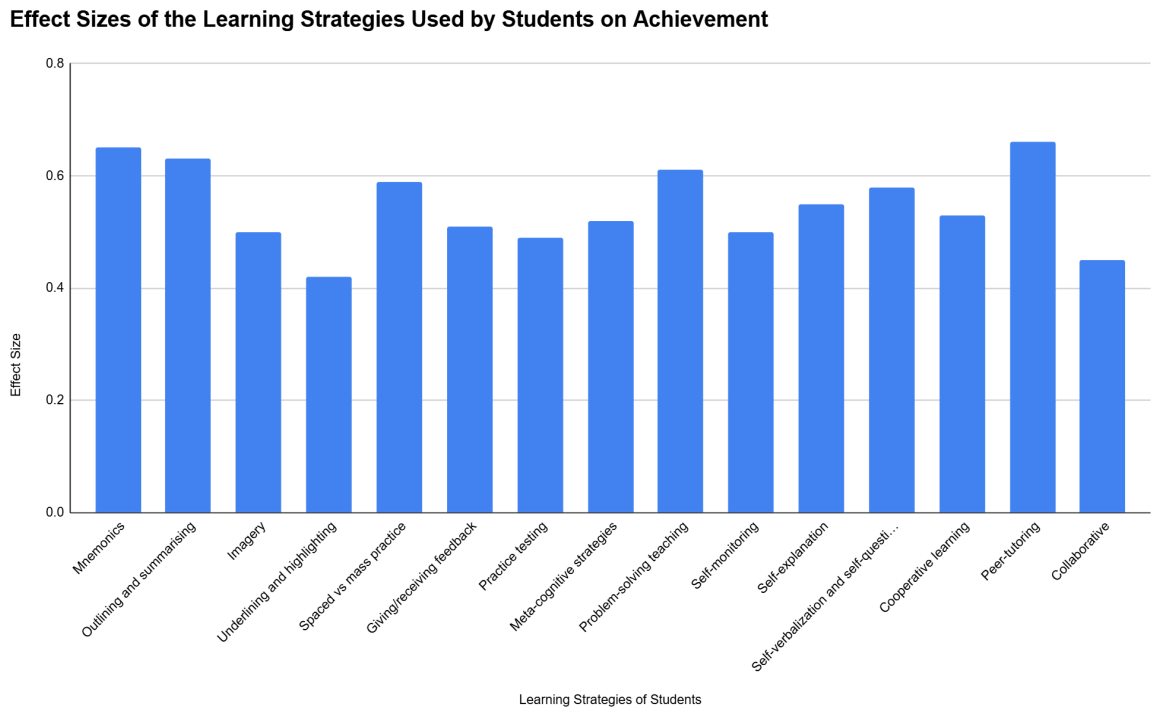


Table 2.7: Effect sizes of the achievement for learning strategies used by students

Strategy	Weighted effect size	Robustness
Mnemonics	0.65	3
Outlining and summarising	0.63	5
Imagery	0.50	3
Underlining and highlighting	0.42	3
Spaced vs mass practice	0.59	4
Giving/receiving feedback	0.51	4
Practice testing	0.49	5
Meta-cognitive strategies	0.52	5

Problem-solving teaching	0.61	5
Self-monitoring	0.50	3
Self-explanation	0.55	3
Self-verbalization and self-questioning	0.58	4
Cooperative learning	0.53	5
Peer-tutoring	0.66	5
Collaborative	0.45	4

A bar graph of the effect sizes of the learning strategies used by students on achievement is given below:



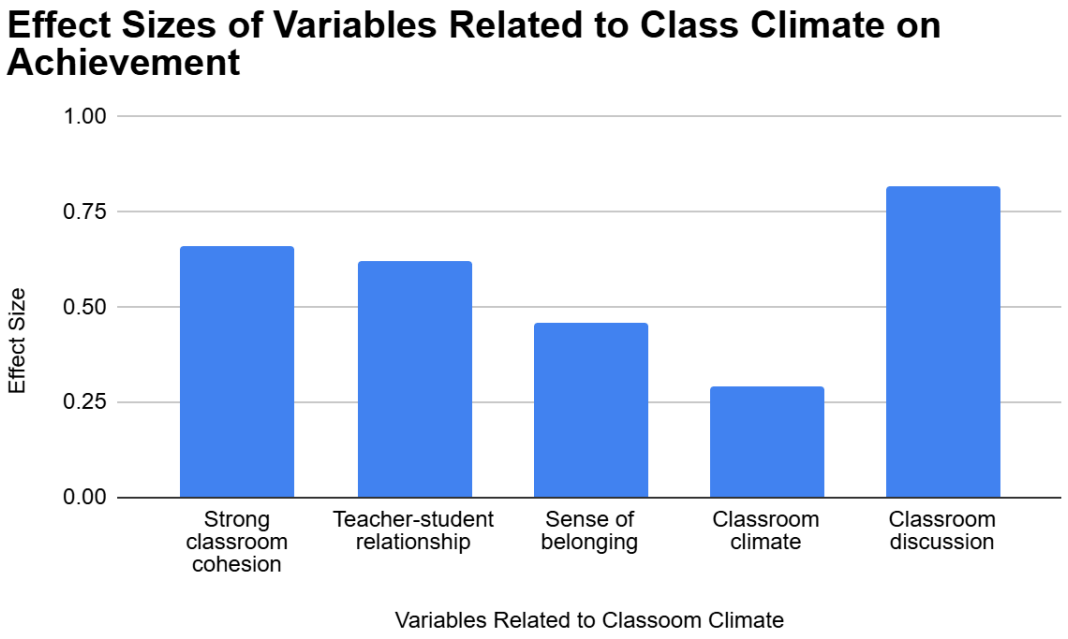
Data on Classroom Climate

Students' achievement also depends on the classroom climate. The effect sizes of the effect of variables in classroom climate on achievement are given in Table 2.8.

Table 2.8: Effect sizes of variables related to class climate on achievement

Characteristic	Weighted effect size	Robustness
Strong classroom cohesion	0.66	3
Teacher-student relationship	0.62	5
Sense of belonging	0.46	3
Classroom climate	0.29	4
Classroom discussion	0.82	1

A bar graph of effect sizes of variables related to class climate on achievement is given below:



Most of Hattie's studies were carried out in developed countries, and his study included studies on all subjects. It included only 7 studies from India.

The evidence from meta-analyses of studies on the effectiveness of different variables on maths achievement is given while discussing those variables.

3

Important Strategies and Traits for Learning Mathematics

Strategies for Learning Mathematics

Insights from cognitive psychology indicate that students learn best when they are engaged in learning, retrieve it after learning, use spaced practice and interleave topics to store it in long-term memory, and use feedback-driven metacognition. (Agarwal, P. K. & Bain, P. M., 2019). Furthermore, they need a deep understanding of concepts to consolidate them in long-term memory for retrieval when needed later.

Engagement

When students are engaged in learning, they are more likely to retain information, grasp the underlying ideas, and solve problems by retrieving relevant information and using appropriate strategies.

Research shows that when you stay focused on learning, you learn more effectively and quickly (Fox, D. et al., 2009).

Lei, H. et al. (2018), in a meta-analysis, found all three engagement dimensions (behavioural, emotional, and cognitive) positively predict maths achievement, with effect sizes ranging from small to moderate ($r = 0.20 - 0.35$). Cognitive engagement (e.g. metacognitive strategies, persistence) had the strongest effect on maths performance. The relationship was stronger in secondary school compared to primary school.

Schneider, M., & Preckel, F. (2017), in a systematic review of meta-analyses, found that interventions targeting engagement (e.g. growth mindset, active learning) significantly improve maths performance (average effect size: $d = 0.42$).

Tze, V.M.C., et al. (2016), in a meta-analysis, found boredom (a form of disengagement) has a moderate negative effect on maths performance (r = -0.24). The negative impact is stronger in maths compared to other subjects, possibly due to its cumulative skill-based nature.

Hattie's effect size for engagement was 0.41 with a robustness of 5

How teachers can ensure engagement in the classroom

- Start with a **hook**: A "hook" is a captivating introduction that piques students' curiosity and interest. It should be short and novel. It can be from the history of maths, a real-world application, a thought-provoking question, a surprising fact, a captivating story, a game, or a puzzle. Ensure the hook is related to the mathematical concept you're teaching.
- Ask questions, e.g., how do we use numbers in daily life?
- Talk about or show the ubiquitousness of the *Golden Ratio* in nature, architecture, and art.
- Talk about the concept of the number zero (contributed by India) and its role in evolving the most useful number system.
- Pose a challenging or intriguing question: This can spark curiosity and encourage students to think critically. For example, if you could fold a piece of paper in half 4 times and cut along the folds, how many pieces of paper would you have?"
- Share a surprising or counterintuitive mathematical fact: this can grab students' attention and make them want to learn more. (e.g. showing that the probability of a person having a rare disease given that he/she tested positive is much smaller than that suggested by the accuracy of a test). Tell a story related to the mathematical concept, which makes the material more relatable and memorable.
- Communicate the learning objectives and expectations for each lesson. This helps students to stay focused on the task.

- Provide for discussions among students and between teachers and students.
- Use engaging instructional strategies such as the concrete, representational, and abstract (CRA) approach and application to real-world problems.
- Give low-floor and high-ceiling tasks so that all students can achieve a level of success and advanced students are not bored.
- Model enthusiasm and enjoyment of mathematics.
- Provide specific and timely feedback on students' work. This can help them stay motivated and focused on improving their understanding.
- Incorporate short breaks to help students rest their minds and return to work with renewed focus.
- Promote mindfulness practices such as deep breathing or meditation to help students improve their focus and attention.

Retrieval

Retrieval involves bringing information to mind from your memory. It is more effective than rereading, underlining, highlighting, or taking notes for long-term retention. It also helps learners to understand what they know and what they don't, leading to improved metacognitive awareness.

The benefit of practising retrieval has been shown to work in many different contexts and with different types of learners, both in the lab and in live educational settings (e.g. Adesope, O. O. et al., 2017; Agarwal, P. K. & Bain, P. M., 2019; Oakley et al., 2021; Roediger, H. L. III, et al., 2014; Sarma, S. & Yoquinto, L., 2020).

Spaced practice

Spaced learning means distributing your learning sessions over time rather than a big blocked practice of each. Research has shown that spaced learning is more effective in the long run than blocked practice (e.g. Agarwal, P. K. &

Bain, P. M., 2019; Sarma, S. & Yoquinto, L., 2020; Roediger, H. L. III et al., 2014).

Hattie's effect size for spaced vs mass practice was 0.59 with a robustness of 4.

Interleaving

Interleaving involves learning or practising multiple related skills and topics simultaneously while alternating between them. By switching between related concepts, interleaving promotes the ability to discern key differences between concepts and enhances the ability to apply and transfer knowledge to new and varied situations. It enhances learning (e.g. Agarwal, P. K. and Bain, P. M., 2019; Sarma, S. and Yoquinto, L., 2020; Roediger, H. L. III, et al., 2014)

For more on interleaving, see How to Use Interleaving for Deeper Learning | Coursera

Feedback-Driven Metacognition

Feedback-driven metacognition refers to the process where students assess their learning after receiving feedback that informs them about their knowledge gaps. It enhances metacognitive skills in helping students to make judgements about their understanding.

For more on the incorporation of retrieval practice, spaced learning, interleaving and feedback-driven metacognition in classroom settings and evidence of its effectiveness in classroom settings, see Agarwal, P.K. & Bain, P.M., 2019; Sarma, S. and Yoquinto, L., 2020; Brown, P. C. et al., 2014.

Deep Processing of Concepts

Deep understanding of concepts involves

- Multiple representations of concepts using manipulatives, visuals, tables, words and symbols.

- Relation of concepts to concepts learnt earlier. For main ideas and connections between them for different classes, see Boaler, J. (2023).
- Explaining the concept to others.
- Application of concepts to real-world scenarios and other subjects.
- Recognition of patterns formed by using the concept.
- Using concepts to develop divergent thinking and inductive, deductive and spatial reasoning.
- Application of concepts to solve problems in real-life situations and novel problems (rich questions).

A more detailed discussion and assessment of deep processing of concepts is given in Chapters 5 and 6, respectively.

Traits in the Affective Domain that Enhance or Hinder Mathematics Learning

The affective domain refers to the emotional side of learning. It encompasses emotions, attitudes, values, beliefs, interests, and motivations that influence how students approach and engage with maths. These are just as important as the variables in the cognitive domain. NEP (2020) has also recommended their development.

Research in psychology has identified many traits in the affective domain that enhance or hinder learning. To foster student success in maths, it's essential to cultivate these positive traits and tackle the negative ones.

Traits that enhance learning

The important traits that enhance maths learning are motivation, curiosity, growth mindset, perseverance, and self-efficacy.

Motivation

Motivation is an internal drive that gets us pushing forward, even when things get tough. Motivation is important as

- It provides the focus, perseverance, and energy needed to overcome obstacles and stay on track.
- Being motivated and actively pursuing goals can lead to more happiness, satisfaction, and a sense of accomplishment. It also contributes to a feeling of control over your life and boosts self-confidence.
- It fuels the curiosity and desire to explore new things, develop new skills, and engage in lifelong learning.
- It allows you to channel your energy effectively and strive for excellence in academics, sports, or your career.

A meta-analysis by Chik, P. Y., & Tay, L. H. (2018) confirmed the strong association between motivation and mathematics performance.

The effect size in Hattie's study for deep motivation was 0.58 with a robustness of 5.

Strategies that would be helpful to spark motivation

Children are motivated best by different motivational forces. Lavoie, R. (2008) lists the following motivational forces: gregariousness, affiliation, achievement, aggression, status, power, inquisitiveness, and autonomy.

To motivate all students

- Utilise a wide variety of approaches, strategies and techniques.
- Teach enthusiastically.
- Focus on the strengths of students to enrich the relationships between students and teachers.

- Recognise, reinforce, and celebrate success, effort and progress.
- Establish specific, measurable, achievable, relevant, and time-bound goals to provide a clear roadmap and a sense of direction, and monitor and assess the progress toward these goals.
- Divide big goals into smaller, manageable steps. Completing mini-goals gives students a sense of accomplishment which keeps them motivated to move forward.
- Demonstrate that you genuinely care for your students and their progress by making each student feel that he/she is valued and recognised as an integral part of the class. Individual feedback on their learning and how to improve it can promote it.
- Promote peer relationships by providing opportunities to work together and celebrate cooperation and collaboration.
- Provide for individual differences by matching the challenge of the question to the student's mastery in the area and giving low-floor and high-ceiling tasks.

Curiosity

Curiosity is an insatiable drive to learn and explore. Curiosity is valuable as

- It can contribute to a positive attitude towards maths, reducing maths anxiety and increasing enjoyment.
- It propels students to ask questions, delve deeper into subjects, and discover new things, keeping their minds sharp and engaged.
- A curious mind is constantly seeking new perspectives and connections. This open-mindedness fosters creative thinking and the ability to approach problems from different angles, leading to innovative solutions.
- It compels students to dig beneath the surface of information. By actively questioning and seeking clarification, they gain a deeper understanding.

- It is a source of joy and fulfilment. As students can experience the wonder of discovery and the satisfaction of learning something new. This can contribute to a greater sense of happiness and well-being.
- It fosters genuine interest in others. By asking questions and actively listening, students can build stronger relationships and connect with people on a deeper level.

Shah, P.E. et al. (2018) established curiosity as an important, independent contributor to early academic success in both reading and math, with a particularly strong and beneficial impact on children from lower socioeconomic backgrounds.

Jirout, J., & Klahr, D. (2012), in a review of studies on inquiry-based learning and curiosity in maths/science, found that fostering curiosity (e.g. through open-ended questions) improves conceptual understanding in maths.

The effect size in Hattie's study for curiosity was 0.74 with a robustness of 2.

Strategies to ignite maths curiosity

- Foster a classroom culture of curiosity and exploration.
    - Introduce puzzles, riddles, and games that require logical thinking and problem-solving.

      For more on these without the use of technology, see

        - Gaskins, D. 2021 and 2023
        - https://nrich.maths.org/5795
        - https://www.solutiontree.com/nurturing-maths-curiosity-with-learners-in-classes-k-2.html
        - https://mathsforlove.com/lessons/rich-tasks/
    - Provide opportunities for hands-on activities and experiments.

- Encourage divergent thinking by giving open questions with many correct answers or many strategies for solving them, followed by discussion.
- Provide opportunities to discover patterns and relationships in numbers and figures.
- Link maths concepts to real-life situations.
- Promote peer interaction and knowledge sharing through discussion and group work.

For more on how to nurture curiosity in elementary school children, see Rumsey, C., Guarino, J. (2024 & 2025).

Mindsets

According to Dweck, C. (2006), people have two mindsets: growth and fixed.

Fixed mindset

People that have a fixed mindset believe that a person is born with a certain amount of intelligence and it can't be changed. According to her, such persons

- Avoid challenges.
- Give up if the task is difficult.
- See effort as fruitless.
- Ignore negative feedback.
- Feel threatened by the success of others.

As a result, they achieve less than their full potential

Growth Mindset

Persons that have a growth mindset believe that a person's talents and abilities can be developed over time through effort and persistence. Persons with a growth mindset

- Embrace challenges
- Persist if the task is difficult.
- See effort as the path to mastery
- Learn from criticism
- Find lessons and inspiration in the success of others

As a result, they reach high levels of achievement.

In a conversation with Boaler, J. (2019, p. 103 of ebook), Carol Dweck indicated that now she thinks that people may have different mindsets at different times and in different situations, but then they need to recognise when they have a fixed mindset.

Sisk, Victoria F. et al. (2018), in a meta-analysis of 273 studies involving 365,915 students, examined the strength of the relationship between mind-set and academic achievement and potential moderating factors. Overall effects were weak for the meta-analyses. However, some results supported the findings: students with low socioeconomic status or who are academically at risk benefit from mind-set interventions.

The effect size for growth mindset in Hattie's study was also low: 0.19 with a robustness of 4.

However, most scholars and research on brain plasticity in neuroscience recommend it.

Development of mindsets

Mindsets are developed by

- Labelling children as smart.

  A study by Gunderson, E.A. et al. (2013) showed that the praise given by parents to 14-38-month-old children predicted the mindset they had when they were 7-8 years old.

  One of the studies by Dweck, C. (2007) revealed the immediate impact of the word “smart". Two groups of students were given a challenging task. On completion, one group was praised for being “really smart”, and the other was praised for “working hard”. Both groups were then offered a choice between two follow-up tasks; one was described as challenging, the other easy. Ninety per cent of the students who were praised for working hard chose the challenging task, and the majority of students who were praised for being smart chose the easy task.

- Belief in stereotypes such that girls or students from disadvantaged backgrounds are not good at maths also leads to fixed mindsets.

  Boaler, J. (2022) reported that just having students check boxes about sex and race was enough to invoke internalised stereotypes that affected test performance.

However, the good news is that mindsets can be changed.

As evidence, Dweck, C. (2019) cites a large single study of growth mindset to date, in which more than 12,000 ninth graders across the nation were randomly assigned to receive a mindset boost. Though it primarily benefited low performers, even high-achieving students who watched short online lessons in ninth grade were more likely to take advanced maths courses in 10th grade than high achievers who didn’t watch the videos.

Development of a Growth Mindset

To develop a growth mindset, teachers should

- Tell students that the brain can change and grow with practice; thus, we can develop abilities and intelligence through dedication and hard work.
- Instil in students that struggle is good and mistakes are opportunities to learn and grow. Moreover, when we struggle, our brain pathways form, connect and strengthen.
- Tell students if they find something difficult, think I have not learnt it yet instead of thinking I will never learn it.
- Provide feedback to help students, which helps students to identify areas for improvement and learn from their mistakes.
- Praise students for their effort, persistence, and problem-solving strategies, which they can control and improve over time, not natural talent. Carol Dweck now thinks that the praise should be limited to the effort that leads to something important (Boaler, J., 2019, p. 103). However, a teacher can praise the fact that correct thinking was used for part of it or that effort led to some result that could be used for moving forward.
- Provide a classroom environment where students feel safe and valued and feel comfortable asking questions and taking risks.

Self-efficacy in mathematics

Self-efficacy refers to a student's belief in their ability to succeed in maths tasks and challenges. Students high on self-efficacy are more likely

- To believe they can learn the concepts being taught and solve the problems presented.
- To be motivated to put in the effort to learn maths.
- To persist in the face of obstacles and try different strategies.
- Have less maths anxiety.

Yang, Y. et al. (2024) conducted a study utilising data from the 2019 National Assessment of Educational Progress (NAEP) to investigate the association between mathematics self-efficacy, gender, race/ethnicity, and school level (e.g. school location, the proportion of under-represented students) and maths achievement to provide a comprehensive analysis of the factors contributing to mathematics achievement among students in classes 4, 8, and 12. In this study self-efficacy emerged as a robust predictor of students' mathematics achievement, exhibiting significant effect sizes for classes 4, 8, and 12. Remarkably, when students' mathematics self-efficacy was held constant, the mathematics achievement gaps among different student subgroups by gender, race/ethnicity, ELL (students who did not learn English as their first and primary language), IEP (Individual Education Program) and NSLP (National School Lunch Program), and socio-economic status narrowed, highlighting the importance of self-efficacy in addressing these disparities.

Huang, Y. (2022) found a moderate to strong effect between self-efficacy and maths performance ($r = 0.38$). He also found that the effect was stronger in older students (secondary/university) than in elementary students.

He also noted that self-efficacy was a stronger predictor of maths performance than general motivation or anxiety.

Richardson et al. (2020), in a meta-analysis, found an $r = 0.42$ between self-efficacy and maths achievement.

He also found self-efficacy had a larger impact than self-concept ($r = 0.35$) and growth mindset ($r = 0.18$), and the effects were stronger in western countries than East Asian contexts.

Honicke & Broadbent (2016), in a longitudinal meta-analysis, found an $r = 0.31$ for the cross-sectional study and $r = 0.24$ for the longitudinal study, showing a slight decrease over time.

Sitzmann, T., & Yeo, G. (2013), in an intervention meta-analysis, found an effect size of $d = 0.45$ for interventions boosting self-efficacy that lead to improved maths scores.

Mastery experiences (e.g. scaffolded practice) were more effective than verbal persuasion (e.g. encouragement).

The effect size in Hattie's study for self-efficacy was 0.64 with a robustness of 5.

How teachers can help build self-efficacy in math

Teachers can help build self-efficacy if they

- Provide opportunities for students to experience success in maths, even if it's through small wins. Give low-floor, high-ceiling tasks.
- Praise students for their hard work and perseverance, not just perfect results.
- Give problems that have multiple representations and can be solved using many strategies for solving problems in a collaborative setting so that students adept at different learning strategies can experience success and learn from peers about other learning strategies.
- Provide specific, constructive and timely feedback that helps students learn from their mistakes and improve.
- Foster a classroom culture where students feel comfortable asking questions, taking risks, and learning from each other.

Persistence

Persistence is the ability to persist when you face challenges, setbacks, or confusion. This trait is particularly crucial for tackling complex real-world problems that often require multiple attempts.

- It helps students build resilience, enabling them to bounce back from mistakes and failures.
- Struggling with a problem can lead to deeper understanding when students eventually find a solution.

- Perseverance encourages students to explore different approaches and find creative solutions.
- Successfully overcoming challenges boosts students' confidence in their mathematical abilities.
- A persistent attitude can lead to a more positive outlook towards maths.

Credé, M., et al. (2017), in a meta-analysis, found grit (perseverance of effort) had a correlation with academic performance (r = 0.19), including maths achievement. Persistence was a stronger predictor than interest, and effects were slightly stronger in K-12 than in higher education.

Duckworth, A. L. et al. (2019), in a meta-analysis, found r = 0.24 between persistence and maths achievement. Effects were stronger in longitudinal studies, suggesting persistence leads to cumulative gains in maths.

Richardson, M., Abraham, C., & Bond, R. (2012), in a study on psychological correlates of university students' academic performance, found that effort regulation (a persistence-related construct) had a small but significant effect (r = 0.20) on maths performance.

Poropat, A. E. (2014) examined persistence as a facet of conscientiousness and its impact on maths achievement and found that persistence had a stronger effect (r = 0.25) in maths compared to other subjects. Teacher-rated persistence was a better predictor than self-reported persistence.

Tucker-Drob, E.M. et al. (2016), in a meta-analysis of behavioural genetics studies, found that persistence has a heritable component but also responds to environmental influences. Persistence accounted for about 15% of variance in maths achievement beyond IQ.

The effect size in Hattie’s study for persistence was 0.41 with a robustness of 5.

How teachers can develop persistence

Teachers play a crucial role in cultivating perseverance in their students. Here are some strategies to foster this essential trait:

- Break down large goals into smaller, achievable steps. This will help students to stay focused and motivated.
- Provide for active learning. Freeman, S. et al. (2014) found that active learning is critical for persistence.
- Highlight the importance of hard work and practice in achieving success.
- Encourage students to view errors as stepping stones to understanding.
- Teach problem-solving strategies.
- Develop self-efficacy.
- Promote a growth mindset. Yeager, D.S. and Dweck, C. (2012) found that when students believe that personal characteristics can be developed, it promotes resilience in them.
- Assist students when they encounter difficulties, but avoid giving away answers.
- Promote collaborative learning to help students learn from each other.
- Provide specific feedback that focuses on effort, strategies, and improvement areas.
- Help students reflect on their thinking processes and identify areas for improvement.
- Foster a supportive and encouraging learning environment.
- Celebrate students' perseverance and determination in overcoming challenges.

Traits in the affective domain that hinder learning in mathematics

Many traits hinder learning; chief among them are maths anxiety and procrastination.

Anxiety

Maths anxiety is a negative emotional response to mathematics. It has been consistently linked to lower maths achievement scores.

Students with maths anxiety tend to avoid maths-related activities and courses. While maths anxiety can appear in early elementary school, it's common for it to emerge around middle school or junior high, as kids confront more complicated mathematical concepts. It continues through adulthood (Barroso, C. et al., 2021).

In a longitudinal study, Geary D.C. et al. (2023) followed teens in classes seven through nine. Their maths anxiety tended to increase over those years for all kids, as maths lessons became more difficult and increasingly abstract.

A meta-analysis by Zhang, L. (2019) of 49 studies on "The Relationship Between Maths Anxiety and Maths Performance" indicated a robust negative correlation between maths anxiety and performance.

Barroso, C. et al. (2021), in a meta-analysis of 747 studies conducted from 1992 to 2018, found $r = -0.28$ between maths anxiety and maths achievement, starting in childhood and remaining significant through adulthood, and a correlation $r = -0.30$ between maths anxiety and achievement, moderated by age (stronger in older students) and assessment type (worse on timed tests).

Sammallahti, E. et al. (2023), in a meta-analysis of 50 studies and 75 effect sizes, found an average effect size ($g = -0.47$) for anxiety.

Foley, A. E. et al. (2017) found maths anxiety predicts lower performance across 64 countries, with heightened effects in high-achieving environments.

Namkung, J. M., Peng, P., & Lin, X. (2019) found a moderate negative effect (r = -0.34), with stronger impacts on complex problem-solving than basic arithmetic.

The effect size in Hattie's study for anxiety was -0.40 with a robustness of 5.

Development of maths anxiety

Maths anxiety is developed by

- Repeated failures in maths.
- Believing maths is difficult or irrelevant.
- Parents' and teachers' maths anxiety.

Beiblock, S. (2010) in his study on the effect of stress or pressure on the brain found that when people work under stress or pressure, their working memory is impeded. This means when students are given timed tests, they become anxious, and that leads to low performance.

It also revealed that the students whose parents made the statement "I was not good at maths" had lower achievement in school; the amount of knowledge parents had was not important.

Maths anxiety consumes working memory, impairing performance (Ashcraft, M. H., & Krause, J. A., 2007; Zhang, L. et al., 2019).

How to reduce maths anxiety

Teachers can reduce maths anxiety by

- Providing opportunities for success and celebrating achievements.
- Not insisting on speed while teaching maths facts and procedures.
- Emphasising understanding of concepts.

- Using a CRA approach so that students adept at different modes can experience success.
- Teaching effective problem-solving strategies.
- Challenging misconceptions and stereotypes.
- Fostering a supportive and encouraging atmosphere.
- Regulating emotions. Sammallahti, E. et al. (2023) found that interventions that focused on cognitive support or regulating emotions were effective both in reducing maths anxiety and improving maths performance.
- Reframing anxiety as excitement (Bieg et al., 2015).

Procrastination

Academic procrastination is the voluntary and irrational delay of working on academic tasks despite expecting the negative consequences of that delay. Its consequences in maths are more serious, as maths is a hierarchical subject.

Kim, K. R. & Eun H. S. (2015) conducted a meta-analysis of 33 relevant studies involving a total of 38,529 participants. This analysis revealed that procrastination was negatively correlated with academic performance. This relationship was influenced by the choice of measures or indicators. The use of self-report scales interfered with detection of a significant relationship between procrastination and academic performance. The demographic characteristics of participants in individual studies also affected the observed relationship.

The effect size in Hattie's study for procrastination was -0.41 with a robustness of 5.

Why do students procrastinate?

- Fear of failure (e.g. Steel, P. & Klingsieck, K. B., (2016). Some academic tasks (e.g. problem-solving) might lead to negative feelings. To avoid the negative feelings, they seek activities that are more pleasurable at that

moment (e.g. watching TV, playing a video game, or going out with friends).

- Failure of time management (e.g. Steel, P. & Klingsieck, K. B., 2016).
- Failure to plan: Students might think a task will not take much time and leave it to do later and then learn that they underestimated the time the task would take.

Why is it important to overcome procrastination?

People who procrastinate have

- Lower self-efficacy which increases avoidance (Wäschle, K., Allgaier, A., Lachner, A., Fink, S., & Nückles, M., 2014).
- Worse health outcomes, e.g., getting sick.
- Increased levels of depression and anxiety (Peixoto et al., 2021).
- Less time to engage in the most effective learning strategies, like spaced learning and retrieval practice.

How Teachers Can Reduce Academic Procrastination

Teachers may suggest some ways to students to avoid procrastination, e.g.

- Plan to work on assignments or tasks, and force yourself to start even if you only work for a few minutes.
- Use the **Pomodoro technique** (25 minutes of work, 5 minutes of break, and repeat).
- Break large tasks into smaller tasks so they seem more manageable.
- Create a checklist with deadlines for each small task.
- Remove sources of temptation. Put common sources of temptation away and out of sight (e.g. TV remote, video game console, phone).

- Think about your long-term goals. Reflect on your goals for the next couple of weeks or even longer.
- Create a plan for reaching your goals and write them down in a place where you can see them.
- Build academic peer support. Create accountability groups with friends in classes or others trying to overcome procrastination.
- Reward yourselves for staying on track.
- Do not believe in negative stereotypes about mathematics, e.g., maths is a subject only for certain people, typically those who are naturally gifted or have a particular aptitude.
- Time-management training: Sammallahti, E. et al. (2023) found time-management training to be most effective for reducing procrastination in maths learning (d = 0.45).

Interconnectedness of traits

These traits are intricately linked and significantly influence maths learning outcomes. Although students' experiences, cultural factors, and teaching practices can significantly influence the strength of these relationships. However, understanding these interconnections provides valuable insights into student success and challenges in maths education**.**

- Curiosity, growth mindset, perseverance, and self-efficacy are positively correlated and reinforce each other.
- Individuals with a growth mindset are more likely to view mistakes as learning opportunities, which can boost motivation.
- High self-efficacy often leads to increased motivation for maths learning.
- Collaboration can foster a supportive learning environment where students can encourage and motivate each other, leading to increased persistence.

- Effective communication with teachers and peers can enhance motivation by providing support, feedback, and a sense of belonging.
- High levels of maths anxiety tend to decrease self-efficacy and motivation.
- Procrastination can be influenced by low self-efficacy. Individuals who doubt their abilities may avoid challenging tasks.
- Students with high maths anxiety tend to have lower self-efficacy in maths.
- Procrastination can decrease motivation as tasks become more overwhelming.

Key Takeaways

Effective Learning Strategies

Insights from psychology indicate that students learn best when they are engaged in learning, retrieve it after learning, use spaced practice, interleave related concepts, are given feedback that helps in developing metacognition, and have a deep understanding of concepts.

Implications for Teachers

- Use hooks, puzzles, storytelling, and challenging questions to capture attention.
- Employ retrieval practice (e.g., exit tickets, low-stakes quizzes, think-pair-share) for longer retention.
- Incorporate spaced practice and interleave related concepts in homework/classwork for long-term retention and consolidation.
- Give feedback that helps in developing metacognition.
- For deep processing of concepts, employ and ask students to use these strategies:

- Multiple representations of concepts using manipulatives, visuals, tables, words and symbols.
- Connect concepts to concepts learnt earlier.
- Explain the concept to others.
- Apply concepts to real-world scenarios and other subjects.
- Open questions followed by discussion

Traits in the affective domain that enhance learning

The important traits that enhance maths learning are motivation, curiosity, growth mindset, perseverance and self-efficacy.

Implications for Teachers

- Develop motivation by setting specific goals, offering praise for effort, and promoting autonomy.
- Encourage curiosity with puzzles, open questions, and rich tasks.
- Cultivate a growth mindset by telling students about brain plasticity and instilling in students the belief that all of us can learn anything through effort and persistence.
- Praise students for effort and persistence rather than natural ability.
- Boost self-efficacy through small wins, collaborative tasks, and individualised support.
- Develop persistence by promoting a growth mindset and collaborative learning and emphasising mistakes as opportunities for learning and helping our brains grow.

Traits that hinder learning

The important traits that enhance maths learning are maths anxiety and procrastination.

- Tackle math anxiety by avoiding pressure for speed, creating a safe space for learning, and using CRA strategies.
- Address procrastination by teaching time management, breaking down tasks, and encouraging peer accountability.

4

How the Brain Learns Mathematics: Neuroscience Insights for Effective Teaching

Neuroscience is the scientific study of the nervous system (the brain, spinal cord, and peripheral nervous system), its functions, and its disorders.

Brain

The human brain is a highly complex organ responsible for cognition, emotions, behaviour, and learning. Its composition includes various structures, cells, and biochemical processes that work together to control these.

We will only be concerned with the anatomical structure of the brain and cells, processes and the role of different parts or processes in learning maths.

Composition of the Brain

A. Anatomical Structure of the Brain: It consists of the cerebrum, cerebellum, brain stem, limbic system, and ganglia. Figure 4.1 shows the main parts of the brain.

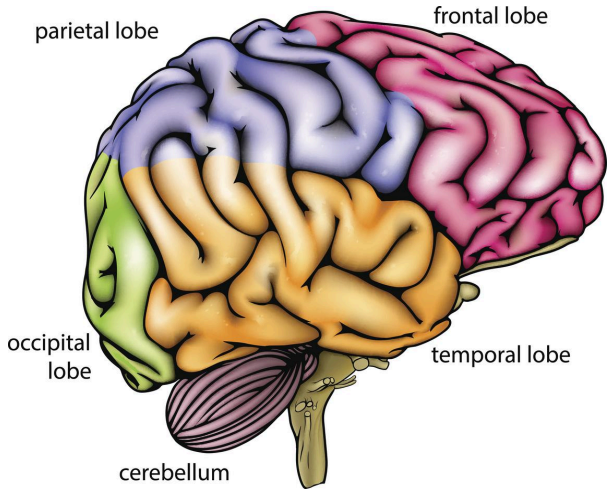


Figure 4.1: Main parts of the human brain (Source: istock/ChrisGorgio)

- **Cerebrum**

  The cerebrum is the largest part of the brain and is divided into the **left hemisphere** and the **right hemisphere**.

  It is responsible for higher-order functions like thinking, reasoning, and learning.

  It is divided into four lobes:

  Frontal Lobe: It is involved in decision-making, problem-solving, and planning, motor control, speech production (Broca's area), and emotional regulation.

  Temporal Lobe: Processes auditory information and is critical for memory formation, language comprehension (Wernicke's area), and emotional association.

  Parietal Lobe: Integrates sensory information and spatial awareness and numerical cognition.

  Occipital Lobe: Processes visual information—recognition of shapes, colours, and motion.

  Cortex: It is the outer layer of the cerebrum that controls higher mental functions such as thinking, memory, language, and voluntary movement. It is made up of **grey matter** and is divided into areas that process sensory information, plan actions, and support reasoning and problem-solving.

  It receives sensory information through the auditory cortex and visual cortex.

  Auditory cortex: It is a part of the temporal lobe that processes sounds, including pitch, volume, rhythm, and speech, and enables us to understand and interpret what we hear.

Visual cortex: It is a part of the occipital lobe that receives and interprets information from the eyes. It helps us recognise shapes, colours, movement, and spatial patterns, making visual perception possible.

Neocortex: It is the largest and most recently developed part of the cerebral cortex and is made up of six layers of neurons. The sensory information after processing is stored here.

It is involved in advanced mental abilities such as reasoning, planning, language, and conscious thought and is especially well developed in humans.

Folds: To increase surface area, it has **gyri** (ridges) and **sulci** (grooves). This folding allows more neural connections within the limited space of the skull.

2.**Cerebellum**: It is located at the back of the brain and coordinates motor movements, balance, and procedural learning (e.g., bike riding).

3.**Brainstem:** It connects the brain to the spinal cord and regulates basic life functions like breathing, heart rate, and sleep.

4.**Limbic System**: The limbic system is located deep within the brain, underneath the cerebral cortex and above the brainstem. It's situated in the cerebrum, the largest part of the brain. It is a group of structures involved in emotion, memory, and motivation. Its key components are given in Figure 4.2.

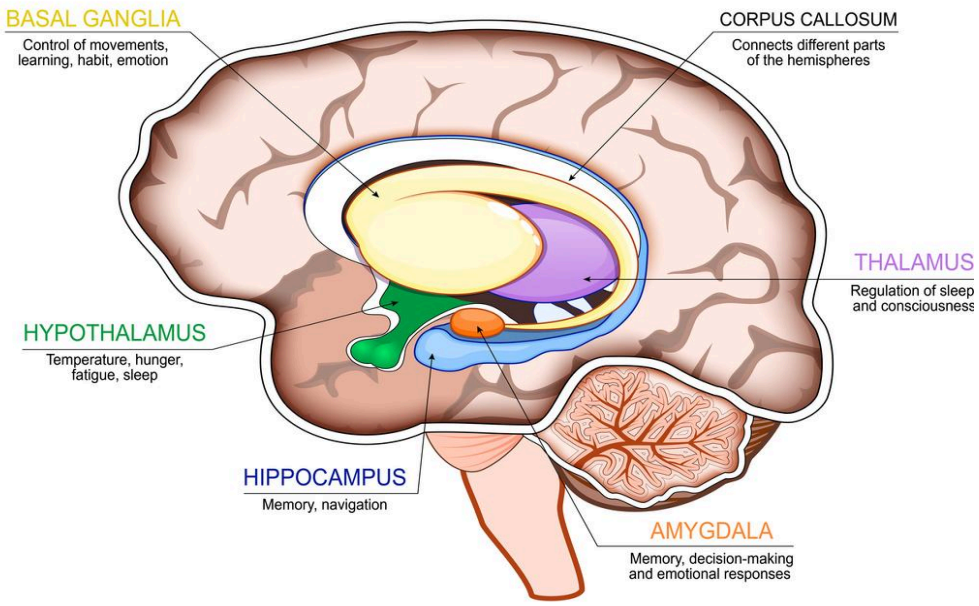


Figure 4.2. Limbic System (istock/Zhabska, T.)

- **Hippocampus**: It plays a key role in forming memories, consolidating short-term memories into long-term memories, and retrieving memories.
- **Amygdala:** It processes emotions, especially fear, anxiety, and pleasure.
- **Hypothalamus**: It regulates hormones and homeostasis.
- **Thalamus:** It acts as a relay station for sensory information.
- **Extended limbic structures:** A few structures are closely connected to the limbic system instead of being involved as primary components. These are responsible for emotional regulation, certain behaviours, pain, and autonomic motor functions.
    - **Basal Ganglia:** It is located deep in the cerebral hemispheres. The primary responsibilities of the basal ganglia include regulating voluntary movements such as eye movements and helping with

balance and posture.
It supports maths learning by automating routines, reinforcing correct solutions, and optimising cognitive effort.

B. Cellular Structure of the Brain: It consists of neurons, glial cells, and schwann cells.

Neurons: Neurones are the building blocks of the brain. The brain has about one hundred billion neurons. Each of these is connected to 10,000 neurons. These are responsible for transmitting information via electrical and chemical signals.

A neuron is composed of dendrites, a cell body, axons, and synapses. Its structure is shown in Figure 4.3

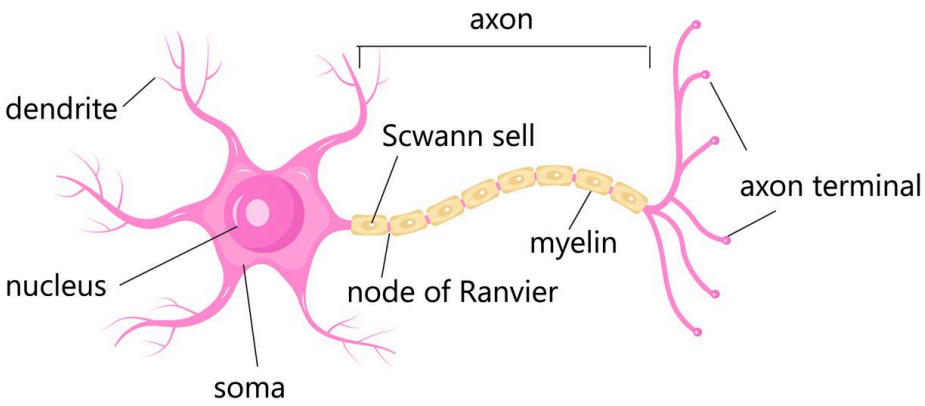


Figure 4.3. Structure of a neuron (istock/Yuliya Baranych)

Dendrites: These receive signals from other neurons.

Soma or Cell Body: It contains the nucleus and is responsible for maintaining cell function.

Axon: It transmits signals to other neurons or muscles.

Synapses: These are junctions where neurons communicate using neurotransmitters.

Myelin sheath: A fatty substance that surrounds and insulates a neuron's axon.

Nodes of Ranvier are located between sheaths of **myelin**. They facilitate speedy electric transmission.

Glial cells: These support and protect neurons.

Schwann cells: These are found in the peripheral nervous system. They play a key role in helping nerves to work efficiently by wrapping around nerve fibres and forming a protective layer called the myelin sheath. This insulation allows electrical signals to travel quickly and smoothly along the nerves.

C. Biochemical Composition: It is composed of

Neurotransmitters: These are chemical messengers that transmit signals across synapses. They modulate the strength and efficiency of neuronal connections. These influence the formation, consolidation and retrieval of memories and regulate various aspects of learning, including attention, motivation, reward processing and speed of information processing (Knightly, R., 2023).

The three important neurotransmitters are

Dopamine: Optimal levels of dopamine are essential for maintaining focus, persistence and pleasure during the learning process. Its levels can be optimised by engaging students in activities that bring joy, satisfaction and a sense of accomplishment, e.g. hands-on activities, discussion and solving rich questions with low floors and high ceilings.

Serotonin: It regulates mood, sleep, appetite, and cognitive functions. An imbalance in it may lead to mood disorders affecting motivation, learning, and emotional regulation. Its levels can be optimised by promoting social interactions and forming positive relations.

Acetylcholine: It is involved in various cognitive functions, including attention, learning and memory. It facilitates the encoding of new information and the consolidation of memories.

2. **Hormones**: These are chemicals released into the bloodstream that influence brain function, e.g. cortisol, which affects memory and stress responses, and oxytocin, which influences social bonding and trust.

3. **Neurotrophins**: These are proteins that support neuron growth and survival, e.g. brain-derived neurotrophic factor (BDNF), which is crucial for synaptic plasticity and learning.

Brain Lateralization

It refers to the specialisation of cognitive functions in the left and right hemispheres of the brain. While both hemispheres work together in most tasks, each side tends to dominate certain functions. This division of labour enhances efficiency in processing information and executing complex behaviours.

- The **left hemisphere** (typically dominant in right-handed individuals) dominates in
    - Speech production (in Broca's area) and language comprehension (in Wernicke's area) are usually left-lateralised.
    - Logical & analytical tasks such as maths, reasoning, and sequential processing.
    - Fine motor control, especially in the right hand.
- The **right hemisphere** is dominant in
    - Spatial abilities, e.g. navigation, mental rotation, and visual-spatial tasks.
    - Processing facial expressions and emotions.
    - Music & creativity, holistic thinking, artistic abilities, and intuition.

- Emotional Processing, e.g. interpreting tone of voice and nonverbal cues.

Each hemisphere controls the opposite side of the body (e.g. the left hemisphere controls right-hand movement), and sensory input from one side of the body is processed in the opposite hemisphere.

Key Cognitive Processes in Learning

Learning how the brain processes, stores and accesses information is essential for optimising learning strategies. It involves **encoding** (transformation of sensory information into neural representations), **consolidation** (stabilisation of information in long-term memory in different regions of the brain) and **retrieval** (accessing information when needed).

Short-term memory: When we are exposed to some information, it is transferred into short-term memory for a short time (a few seconds).

Working memory: After a short time in short-term memory, the information is transferred to working memory. It is involved in holding and manipulating information during maths tasks (e.g. carrying over numbers in addition or solving multi-step problems). The information is encoded in working memory and sent to different regions for processing.

Different types of information are processed in different regions which are activated. *The greater the number of regions activated, the better the learning.*

Attention is crucial for focusing on relevant information and directing it to appropriate regions for processing.

The brain can hold seven pieces of information in working memory and process only 2-4 of these at a time (Sweller, J., 2025): https://open.substack.com/pub/scienceoflearning/p/the-transient-information-e

Overloading memory can hinder learning.

After encoding, information is transferred to long-term memory to be retrieved later.

Long-term memory: The information is **consolidated** here as a set of links between neurons or neural pathways in the brain. It stores mathematical facts, procedures and concepts for multi-step calculations and problem solving. Long-term memories are stored in distributed networks across the neocortex (a part of the cerebral cortex or grey matter).

Emotions and experiences associated with the information can influence its consolidation and subsequent retrieval.

Retrieval involves accessing stored information when needed.

Contextual cues or associations facilitate memory recall by activating the neural networks linked to the stored information.

Retrieval reinforces memory and neural connections.

Neural Pathways

The brain uses two major pathways to store information in long-term memory: declarative and procedural. Each pathway stores its information separately in long-term memory in the neocortex responsible for complex functions like sensory perception, cognition, and language.

The declarative pathway uses working memory and the hippocampus to lay weak links in long-term memory. It is mostly conscious and flexible. It is fast to learn and store information but slower to use.

When you practise what you've learnt, the links get stronger. And the basal ganglia system builds its own direct, automatic sets of links, which let you retrieve answers quickly without conscious thought. This is called the **procedural pathway**.

These two pathways collaborate with the hippocampal system, handling detailed conscious problem-solving, while the basal ganglia's pathways enable rapid, automatic responses.

How the Brain Learns Mathematics

When we are exposed to some sensory input, it

1. Enters through the **visual cortex** and/or **auditory cortex**.
2. The **prefrontal cortex** (the front part of the frontal lobe) directs **attention**, filtering relevant information and sending it to working memory. If attention is scattered, information may not even enter working memory, making learning ineffective.
3. The **prefrontal cortex** (working memory) temporarily holds and manipulates math information. The working memory is small and is easily overloaded, and information can fade away quickly. The working memory works through the hippocampus to lay weak links in long-term memory, scattered around the **neocortex**.
4. When you retrieve and practise with what you've learnt, the links get stronger, and the **basal ganglia system** builds its own direct, automatic sets of links that let you retrieve answers quickly without conscious thought.
5. When you learn concepts related to an earlier concept, the **hippocampus** helps integrate new information with existing links in a neural network.
6. **Amygdala (**The limbic system) influences learning through emotion. Positive emotions, curiosity, and motivation activate reward pathways (dopamine), reinforcing math learning. Negative emotions (e.g., math anxiety) can block working memory and impair performance.

7. Feedback loops between the prefrontal cortex and long-term memory help students monitor what they know and don't know. This supports **metacognition**, enabling learners to adjust strategies, persist through challenges, and refine understanding.

Neuroplasticity

Neuroplasticity refers to the brain's remarkable ability to reorganise its structure, functions, and neural connections in response to learning, experience, injury, or environmental changes. The experiences can strengthen or weaken connections between neurons, form new neurons, reassign functions to undamaged areas after injury or compensate for deficits

How the Brain Develops and its Implications for Learning Mathematics

Brain development is a gradual process that continues from early childhood through adolescence into the mid-twenties. During this time, the brain undergoes structural and functional changes that influence how students perceive, process, and retain mathematical information. Understanding these changes can help educators tailor their instruction to match students' cognitive readiness and learning needs.

A line graph of development of key brain functions over time (from birth to age 25) are shown in Figure 4.4

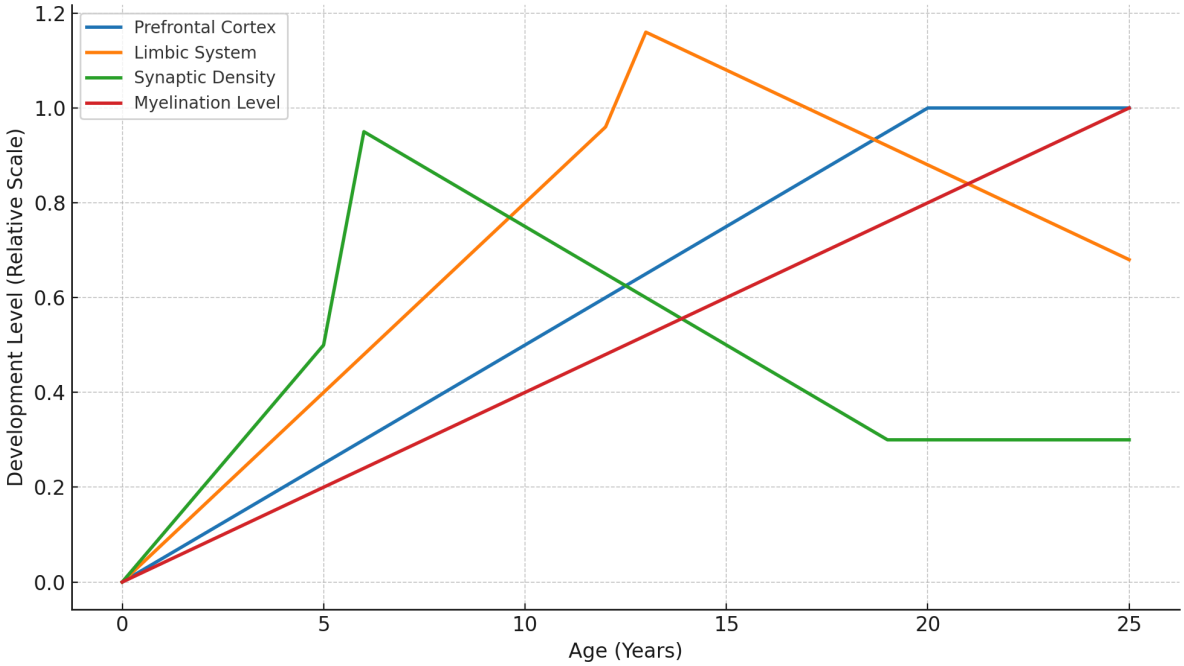


Figure 4.4: Development of brain over time

Source: ChatGPT

- **Prefrontal Cortex** (blue)**:** It is the area of the brain just behind the forehead that controls the planning, decision-making, reasoning, and execution of behaviour and integrates information in working memory.

  It develops gradually, maturing fully by the mid-20s.

- **Limbic system** (orange): It is a group of structures in our brain that regulates our emotions, behaviour, motivation and memory.

  It develops rapidly in early adolescence, peaking before prefrontal maturity.

- **Synaptic Density** (red): It is defined as the number of synapses in a given volume of brain tissue. It is a key indicator of brain development, plasticity and overall cognitive function. n

  It peaks in early childhood, then declines due to pruning.

- **Myelination (**green**)** It is the formation and development of myelin sheaths in the nervous system. This sheath allows nerve signals to travel faster and more efficiently

It keeps growing at a constant rate.

Brain Characteristics in Early Childhood (0–6 years)

- The brain grows rapidly, reaching about 80% of adult size by age 3.
- There is high neural plasticity, and thus experiences significantly shape brain wiring.
- New synapses form rapidly, especially in sensory and language regions.
- Myelination begins, and basic sensory and motor skills develop.

Educational Implications for Math

- Use manipulatives, songs, games and hands-on activities.
- Foster language development to support number sense (e.g., subitising, maths talk, number stories).
- Encourage play-based learning and pattern recognition (shapes, rhythms, sequences).
- Focus on exploration and encouraging curiosity.

Brain characteristics in middle childhood (6-12 years)

- Synaptic pruning begins – unused neural connections are eliminated.
- Working memory and attention span improve.
- Logic and reasoning begin to develop around age 7 (Piaget's concrete operational stage).
- The prefrontal cortex responsible for self-regulation begins to mature.

Educational Implications for Math

- Instil in students the belief that everyone can learn maths through effort and persistence (growth mindset)
- Introduce basic arithmetic using the CRA approach.
- Develop a deep understanding of concepts by elaborating on them.
- Provide for mastery of automaticity of number facts and fluency of procedures.
- Teach problem-solving strategies by explicit teaching.
- Develop communication skills by discussion and writing assignments.
- Provide opportunities for solving rich questions in collaborative settings by encouraging risk-taking and providing a supporting environment.
- Support metacognition by encouraging students to explain their thinking with exit tickets or think-pair-share.

Brain characteristics in adolescence (13–19 years)

The prefrontal cortex responsible for planning, judgement, and impulse control is still developing.

The limbic system has matured, and emotional and reward-seeking behaviours are prominent.

White matter (it is a support tissue that lies beneath the brain's grey matter) has increased, improving communication between brain areas.

There is greater sensitivity to peer influence and emotional reactions.

Implications for Mathematics Education

- Instil in students the belief that everyone can learn maths through effort and persistence.
- Can teach abstract concepts: e.g. algebra, functions, proofs, probability.
- Provide for deep understanding of concepts by using multiple representations and applications of concepts in diverse situations.
- Use real-world math applications that resonate with students' interests.
- Provide opportunities for solving rich questions in collaborative settings by encouraging risk-taking and providing a supporting environment.
- Provide opportunities for metacognition by using exit tickets and journaling.

Primacy-Recency Effect

In a learning episode, we tend to remember best what comes first and second best what comes last (Sousa, D.A., 2022). For example, after reading a grocery list, if you try to recall items, you will remember the first few and the last item or two. This implies that teachers should introduce the new material first, rather than asking students what they already know, as their incorrect responses may be retained. Teachers should not use this time for class management tasks. Sousa, D.A. (2023) also recommended that teachers use downtime for practising new learning or relating it to past learning and divide episodes of learning into sub-episodes of about 20 minutes.

Implications of Research in Neuroscience for Teaching and Learning

The human brain continually organises itself based on experiences. When we learn something, neurons are connected. To enhance learning, we need to strengthen neural connections. The strength of these connections is influenced by the formation of myelin, a fatty substance that wraps around nerve fibres

and enhances the speed and efficiency of neural communication. Neural connections are strengthened by

Practice: When we repeat an activity, another sheath of myelin wraps around the fibres. The greater the layers of myelin sheaths surrounding a fibre, the more efficiently electrical signals are transmitted, making procedural learning more efficient.

This is also corroborated by research in psychology.

Retrieval: Actively recalling information strengthens neural connections, making it easier to access later. Quizzing, flashcards, and self-testing are all effective forms of retrieval practice. It strengthens the declarative pathway.

Pan, S. C., & Rickard, T. C. (2018) found effect sizes d = 0.50 - 0.70 for the effectiveness of retrieval vs restudying. The effects were stronger for procedural maths (e.g. algebra) than conceptual maths.

Yang et al. (2021), in a comprehensive meta-analysis of 222 studies involving over 48,000 students, found that retrieval practice (e.g. quizzing) significantly enhances academic performance across various subjects, including mathematics.

Adesope et al. (2017) found a g = 0.50 for retrieval practice. It increased to g = 0.60 when combined with elaboration.

Geary, D. C. et al. (2012) found that poor retrieval fluency (e.g. slow fact recall) leads to maths difficulties, d = 0.80 - 1.20.

The effect size was largest for arithmetic fluency (d > 1.0).

Jacob, R. & Parkinson, J. (2015) found a multiple correlation of R = 0.30 - 0.40 for predicting maths success from working memory retrieval and inhibition (the brain's capacity to suppress or restrain unwanted thoughts, behaviours, or responses).

Peng, P. & Fuchs, D. (2016) found an r = 0.45–0.55 between working memory retrieval and maths performance.

Verbal working memory is more critical for word problems; visuospatial working memory for geometry.

Hattie's effect size for practice testing was 0.49 with a robustness of 5.

Its effectiveness for learning is also corroborated by classroom research in psychology conducted in classrooms (e.g. Agarwal, P. K. & Bain, P. M., 2019; Brown, P. C. et al., 2014)

Spaced practice: Reviewing material at spaced intervals enhances long-term retention more effectively than cramming just before an examination (e.g. Rohrer, D., 2015; Cepeda, N. J. et al., 2006). It enhances long-term retention.

Hattie's effect size for spaced repetition vs massed practice was 0.59 with a robustness of 4.

Its effectiveness for learning is also corroborated by research in psychology (e.g. Agarwal, P. K. & Bain, P. M., 2019; Brown, P. C. et al., 2014).

Interleaving: Interleaving involves a schedule of homework that mixes questions on multiple related topics, as opposed to the blocked practice of one topic at a time. It strengthens both the procedural and declarative pathways.

Interleaving enhances maths learning more than blocked practice, particularly for transfer and long-term retention (e.g., Rohrer, D., Dedrick, R. F., & Stershic, S., 2015 & 2020; Agarwal, P.K. and Bain, P. M., 2019).

Its effectiveness for learning is also corroborated by classroom research in psychology (e.g., Agarwal, P. K. & Bain, P. M., 2019; Brown, P. C. et al., 2014).

Chunking: The brain can perceive a coherent group of items as a single item or chunk. Thus, chunking information helps to increase the capacity of working memory. It has several advantages, e.g.

- Chunking improves working memory efficiency in maths (Cowan, N. 2001; DeStefano, D. & LeFevre, J., 2004).

- Chunking strategies (e.g. grouping numbers in multi-digit operations) correlate with improved arithmetic fluency ($d \approx 0.45$) (Fuchs, L. S. et al., 2006).
- Chunking improves recall and problem-solving efficiency across domains, including maths (e.g. Gobet, F. et al., 2001).
- Chunking algebraic expressions (e.g. factoring) improves problem-solving accuracy ($d = 0.60$), (Star, J. R., & Rittle-Johnson, B., 2009).
- Chunking is a key strategy for learning abstract maths concepts (Richland, L. E., Stigler, J. W., & Holyoak, K. J., 2012).
- Explicit chunking strategies (e.g. breaking problems into steps) significantly aid students with maths difficulties ($g = 0.72$). (Gersten, R. et al., 2009).

Visualisation: Creating mental images of numbers, equations, or geometric shapes helps students retain and understand information.

Höffler, T. N., & Leutner, D. (2007) found $g = 0.37$ for dynamic visualisations (e.g. animations) vs static images in maths learning, especially for procedural knowledge.

Ainsworth, S. (2006) found a $d \approx 0.50$ for multiple visual representations (e.g. graphs and equations) for conceptual understanding in maths.

Rau, M. A. (2017) found $g = 0.65$ for the use of visual representations (e.g. number lines, diagrams) for problem-solving accuracy in algebra and word problems.

Hegarty, M., & Kozhevnikov, M. (1999) found $r = 0.52$ between schematic visualisations (e.g. graphs) with success in maths problem-solving.

Uttal, D. H. et al. (2013) found $d = 0.47$ for spatial visualisation training (e.g. mental rotation) for geometry performance and transfers to broader maths skills.

Mix, K. S., & Cheng, Y. L. (2012) found r = 0.45 between spatial visualisation ability and maths achievement, especially in geometry.

Gersten, R., C. et al. (2009) found a g = 0.72 for the use of visual aids (e.g. manipulatives, diagrams) for students with maths difficulties

Deeper processing of concepts: It enhances long-term retention, consolidation of concepts, and neural plasticity. It may be attained by

- **Use of the CRA approach for understanding concepts:** Use the concrete, pictorial, and abstract (CrA) approach for multiple representations of concepts by starting with concrete experiences (manipulatives or physical objects) to build understanding, progressing to pictorial representations (pictures, drawings, diagrams) before moving to abstract symbols (numbers and equations) and words. It activates many regions in the brain. The insights from neuroscience suggest that the more the number of regions activated, the greater the strengthening of neural pathways.
- **Developing number sense:** Number sense provides a foundation for maths. It refers to a person's intuitive understanding of numbers, ability to subitise (recognising numbers presented visually without counting), seeing relationships between numbers, flexibility with numbers, and ability to decompose, recompose, and manipulate numbers in various ways, making reasonable guesses about quantities or calculations, and estimating and judging the reasonableness of answers. Teachers can nurture number sense by encouraging mental math, number talks, use of **'ish' numbers** (use of approximate numbers)**,** estimation, pattern recognition, and multiple solution strategies, rather than focusing solely on rote procedures.

  Such practices align with how the brain learns best: through making connections, retrieving knowledge in varied contexts, and strengthening neural networks over time and neural plasticity.
- **Connecting concepts**: Studies have shown that making connections between different mathematical concepts strengthens the neural pathways

for each concept, leading to deeper understanding and better retention of each of the concepts (e.g. Dehaene, S., 1997).

For examples of concept maps for different concepts and classes, see Boaler, J. (2023). For examples of using contexts for applications of many concepts at the same time, see Leinwand, S., 2020.

- **Applying concepts to real-world scenarios and other subjects:** It consolidates new learning with existing neural networks; the networks expand, making the foundational and conceptual knowledge available for more real-world transfer applications.
- **Recognise patterns involving a concept:** Math is essentially the study of patterns. These patterns may be numerical, spatial, algebraic, logical, or procedural.

  Recognising, creating and using maths patterns for problem solving is a powerful driver of neuroplasticity, as it strengthens and extends neural networks.
- **Developing spatial reasoning**: Spatial reasoning refers to the ability to visualise objects and their movements in the mind, understand relationships between objects (e.g. distance, direction, and orientation), manipulate objects mentally or physically (e.g. rotating, flipping, or transforming shapes), interpret and create visual representations (e.g. maps, graphs, and diagrams) and perceive how they would look when viewed from different sides.

  There is a strong link between spatial reasoning and maths comprehension (Mix, K.S. & Cheng, P., 2012). The brain regions responsible for spatial skills are often the same as those involved in visualising geometric shapes, manipulating objects mentally, and understanding spatial relationships.

  Teachers can encourage spatial reasoning by giving tasks which require students to make shapes on a geoboard or dot paper; cutting and folding activities with paper; solving mazes or puzzles to navigate or visualise paths; building objects with blocks or tangram pieces; exploring 3D shapes

by making them; drawing or identifying their nets; identifying cross-sections of 3D shapes when cut by a plane; drawing 3D shapes on paper; and perceiving, manipulating, and mentally visualising spatial relationships and objects in two or three dimensions.

- **Solving rich problems**: A rich problem is a task for which a learner can't see an immediate known solution path. It may be a puzzle, a game, a maths problem, a project or an enquiry task. It presents a high level of cognitive demand and requires an understanding of concepts, carrying out procedures, recognition of patterns, estimation, inductive and deductive reasoning and divergent and critical thinking.

  What is considered a rich task varies with the student's prior knowledge and experiences.

  A rich task will involve many mathematical skills and abilities and hence activate many regions of the brain, e.g. visual areas for processing graphs and diagrams, spatial reasoning areas for understanding geometric concepts, and working memory and problem-solving centres for manipulating information and devising solutions, and thus strengthen the neural networks for all of the skills and abilities.

  Solving rich problems also provides challenge and novelty to ignite students' curiosity and thus increases students' engagement.

  Conditions that may help for engagement and success of students in solving rich problems

  For students to engage and succeed in solving rich tasks, teachers should

  - **Encourage a growth mindset** by telling them about neuroplasticity, that mistakes are a part of learning and these help the brain to grow, and praise effort and persistence rather than innate ability.
  - **Provide for working in collaborative settings:** It provides for social interaction, which activates reward centres in the brain, leading to more engagement and better learning outcomes. Moreover, students can learn

from peers and develop the social skills necessary for working in groups.

- **Encouraging metacognitive strategies**: Metacognition activates several brain regions, notably the prefrontal cortex, which is responsible for executive functions like decision-making, planning, and problem-solving, and the parietal lobe, which is involved in analysing problems and identifying patterns and sequences in numbers, shapes, or other data.
- **Creating a supportive environment to reduce anxiety**: Neuroscientific research shows that maths anxiety activates the amygdala (involved in fear responses) and reduces activity in problem-solving areas of the brain. It can also impair the prefrontal cortex (responsible for reasoning) and impede working memory (Beiblock S., 2010).

  Providing a supportive environment, encouragement of a growth mindset, mindfulness practice and reducing emphasis on speed can help alleviate anxiety.

- **Providing immediate feedback**: The brain learns best when it receives immediate feedback on performance. When students make mistakes, prompt corrective feedback helps them adjust their thinking and strengthen their neural connections.

Individual Differences in Brain and Learning

The variability in working memory capacity and varying strengths and weaknesses in the brain regions of students may affect students' learning of maths.

For students with limited working memory, teachers may

- Break down tasks into smaller, manageable steps.

- Do not insist on speed.
- Use CRA approach.
- Pre-teach vocabulary so language doesn't become a barrier.
- Provide scaffolding.

For students with dyscalculia (difficulty with number sense and arithmetic), dyslexia (difficulty with reading) or ADHD (challenges with attention and working memory practice), targeted interventions can improve maths skills.

Carnegie Learning is wholly devoted to using neuroplastic research. (https://www.scilearn.com/) to help students with learning difficulties. Its programme **Fast Forward** has helped struggling readers achieve their full learning potential. It targets the reading skills of phonemic awareness, phonics, the alphabet, vocabulary, grammar, fluency, and comprehension. It has been shown not only to improve reading and speaking to normal levels after only 6 weeks of training but also to result in strong cognitive skills like memory, attention, processing, and sequencing.

Barbara Arrowsmith has a programme of exercises for a young brain (Barbara Arrowsmith Young Brain Exercises). It aims to treat learning disabilities by identifying and strengthening weak brain capacities. It includes exercises to strengthen 19 areas of cognitive function, such as reasoning, comprehension, maths, reading, visual memory, speaking, writing, vocabulary, and auditory memory.

Key Takeaways

How the brain learns maths

There are billions of neurons in the brain. When we are exposed to some information, it goes to short-term memory for a few seconds and then to working memory, where it can be manipulated. But working memory is limited and can hold a maximum of 7 pieces of information**.** So the

information is encoded and stored in long-term memory, where some neurons are linked. These links are weak and need to be strengthened and consolidated in long-term memory.

Implications for teachers

Teachers can overcome the problem of limited working memory and strengthening the neural links by

- Breaking problems into smaller, manageable steps to overcome the limited capacity of working memory.
- Providing for understanding, retrieval, practice, spaced practice, and interleaving for strengthening and consolidating the neural links in long-term memory.

Multiple brain regions are involved in math learning

When we learn maths, many regions in the brain are activated. For example,

Intraparietal Sulcus for quantity and number sense

Prefrontal Cortex (the front part of the frontal lobe) for working memory, attention and planning

Angular gyrus (a part of parietal lobe) for maths symbols and retrieval of maths facts

Hippocampus for memory consolidation

Basal Ganglia for procedural learning and fluency

Visual and spatial areas for diagrams, graphs and geometry.

The greater the number of regions activated, the better the learning.

Implications for teachers

Provide for activities that activate many regions, e.g., multiple representations of concepts, deep processing of concepts, or rich questions.

Neural pathways

When you first learn something new, your working memory works through the hippocampus to lay weak links in long-term memory. This is called the **declarative pathway.**

When you **practise** with what you've learnt, the links get stronger, and the basal ganglia system builds its own direct, automatic sets of links that let you retrieve answers quickly without conscious thought. This is called the **procedural pathway**. These automatic links provide instant answers without thinking about it, e.g. $5 \times 9 = 45$.

These two pathways collaborate for handling detailed conscious problem-solving, while the basal ganglia's pathways enable rapid, automatic responses; the declarative pathway pays attention to other aspects of problem-solving.

It is important to build automaticity of number facts and fluency of procedures through practice so that students can pay more attention to other aspects of problem solving.

Visual and Spatial Thinking Enhances Math Learning: The brain’s visual pathways are powerful tools for reasoning, estimation, and problem-solving.

Implications for teachers

Use pictures, diagrams, number lines, bar models, area models, graphs, and apps such as GeoGebra and Desmos.

Neuroplasticity of the brain

The brain's ability to change and grow in response to experience implies that all students are capable of learning mathematics if teachers give the right messages and provide the right experiences.

Implications for teachers

- Believe and instil in students the belief that math ability can be developed through effort and perseverance, whatever the innate talent, i.e., develop a growth mindset.
- Instil in students that making mistakes is good, as it provides opportunities for learning and helps our brains grow.
- Give feedback praising effort, strategies, and improvement, and use language that promotes growth: *"You haven't mastered this yet"* rather than *"You're not good at maths."*
- Give low-floor and high-ceiling questions in collaborative settings so that students at different levels of proficiency in the area are engaged.

Language Supports Mathematical Understanding

Language is essential for reasoning and communicating in mathematics.

Implications for teachers

Use mathematical discussions and ask students to write answers to open questions, rich questions solved in groups, and write on questions requiring metacognitive thinking in journals.

The Brain Develops Gradually With Age

Brain development is a gradual process that continues from early childhood through adolescence into the mid-twenties. During this time, the brain undergoes structural and functional changes that influence how students perceive, process, and retain mathematical information.

Key features of brain development and its implications for maths education for different age groups are given in Table 4.1.

Table 4.1: Key features of brain development and its implications for maths education for different age groups

Age group	Brain Development Features	Implications for Math Education
0–6	Rapid synapse growth, high plasticity	Use concrete experiences, math talk, games, and exploratory activities
6–12	Synaptic pruning, growing logic and memory	Emphasise conceptual understanding, patterns, structured routines
13–19	Emotional sensitivity, prefrontal cortex still developing	Provide for metacognition, real-world relevance, collaborative learning

Math anxiety affects mathematical thinking

The brain's emotional centres are deeply linked with learning. **Math anxiety** can activate the brain's fear response, disrupting concentration and memory. A sense of failure or fear of being wrong can shut down learning.

Implications for teachers

Teachers can reduce anxiety by

- Creating a safe space for learning.
- Normalising mistakes as part of learning.
- Avoiding speed pressure for number facts and procedures.

- Using CRA strategies so that students adept at learning through different modes can succeed.
- Praising effort and reasoning, not just correctness.

Primacy-recency effect

In a learning episode, we tend to remember best what comes first and second best what comes last.

Implications for teachers

Teach the new material first and do not ask students what they already know, as their wrong responses may be retained.

Importance of other Factors for Learning

There are many other factors, e.g., attention and focus, motivation, stress, physical exercise, sleep, and technology, that affect learning. For their effect on the brain and strategies to boost or tackle them for optimising learning, see Knightly, R., 2023.

5

Educational Objectives in Mathematics and Teaching Strategies for Enhancing Their Achievement

The National Education Policy (1986) expressed concern about the main emphasis in maths learning being on rote memory and performing procedures and the need to promote higher-level thinking. This concern was again reiterated in NEP (2020) and the National Curriculum Framework for School Education (2005, 2023).

In this chapter, taking the National Curriculum Framework for School Education (2023) as a framework, we elaborate on some approaches that the research suggests are promising and practical in India.

There is no instructional strategy that works in all situations for all objectives and all students, but there is compelling evidence for certain strategies or approaches to have a greater likelihood of helping students attain their learning goals.

Objectives of Mathematics Education

The objectives for learning maths may be classified into

- understanding of concepts.
- The knowledge of facts and maths vocabulary.
- Fluency in procedures
- Communication skills
- Development of abilities for higher-order thinking

Before considering the approaches for teaching for attainment of these objectives, we need to know about the quality of teachers and resources in Indian schools.

Resources in Indian Schools

In India, there is great variation in the quality of teachers and resources from school to school, depending on their location, management, district, and state. About 9.5% of the government schools in India are single-teacher schools; their range in different states and union territories is from 0% to 29% (UDISE+, 2021).

Tata Institute of Social Sciences (TISS), (2023), based on a study in eight states, reported that 38% of the government and 19% of the private schools were multigrade schools.

There are also some excellent public schools with good resources, including digital technologies.

For some data on other facilities in schools at the national level and their range in states and union territories, see Appendix 1.

The Availability and Quality of Teachers

The qualifications, quality and availability of teachers, and teaching and assessment practices were discussed in chapter 1.

To recapulate

- There is a shortage of teachers.
- A large proportion of those qualifying to become do not have adequate mathematics knowledge, as reflected in the mean marks of 46% and 50% in maths component for primary and secondary teachers in Teacher Eligibility Tests (TISS, 2023).
- Overall, only about 46% of teachers teaching in primary classes have a Diploma in Elementary Education, a Bachelor of Elementary Education or equivalent qualifications (TISS, 2023).

Teaching Practices in Mathematics

In most of the schools

- The teachers plan their lessons mainly based on textbooks.
- Students have access to the textbook, notebooks, pencils and pens.
- The students sit in straight rows, with one or more spaces between desks that seat two or more students, listen to the teacher's lectures, take notes, and watch demonstrations. Talking is limited in class.
- Emphasis in teaching is on the memorisation of facts, carrying out procedures, and routine applications.
- Homework is usually done by answering questions in the textbook. The students often focus on the lecture, the class notes, and homework as preparation for the tests, midterm and annual examinations.

Assessment Practices in Mathematics

- Students' learning is measured with paper-and-pencil assessments. All the tests are made by the teacher except for the final examination in class 10, which is conducted by a central or a state board of secondary education.
- Assessments mainly test facts, procedures and routine applications.
- Students receive a grade on assessment in tests, and generally, the next learning segment is introduced without much regard to the level of mastery.

Effective Teaching Strategies for Different Objectives

Deep understanding of concepts

We must first develop a deep understanding of concepts as corroborated by NEP (2020) and research in education, psychology, and neuroscience before

teaching skills, procedures, and applications to real life and other subjects and solving rich problems.

Deep understanding of concepts involves

- Multiple representations of concepts using manipulatives, visuals, tables, words and symbols.
- Relation of concepts to concepts learnt earlier.
- Explaining the concept to others.
- Application of concepts to real-world scenarios and other subjects.
- Recognition of patterns formed by using the concept.
- Use of concepts in estimation and development of intuition, number sense, divergent and inductive, deductive, and spatial reasoning.
- Use of concepts in solving rich problems.

Schneider, M. et al. (2017) found an average effect size of $g = 0.50$ for conceptual and procedural combined vs procedural only and concluded teaching maths with conceptual understanding improves transfer and problem-solving more than rote procedures alone.

Bisra, K. et al. (2018) found an average effect size of $d = 0.55$ for self-explanation (students explaining steps or concepts) in STEM, including maths.

Self-explanation when paired with worked examples was more effective ($d = 0.70$) than self-explanation paired with problem-solving ($d = 0.40$).

Jitendra, A. K. et al. (2015) found a $g = 0.70$ for word problem solving. The effect was strongest for struggling students ($g \approx 0.85$) and elementary/middle school maths ($g \approx 0.75$).

Dunlosky, J. et al. (2013) found an effect size $d \approx 0.40–0.60$ in maths for elaborative interrogation (asking "why" questions).

It was more effective for learners with some prior knowledge.

Rau, M. A. (2017) found a g = 0.65 for visual representations and conceptual explanations in maths. It was best for fractions, algebra, and geometry.

Hattie's effect size for task complexity was 0.62 with a robustness of 4.

Effective teaching strategies for understanding maths concepts

1. Use CRA Approach

It involves beginning instruction with hands-on activities and manipulatives to provide a concrete foundation for understanding and gradually transitioning from concrete experiences to pictorial representations and then to words and symbols.

Benefits of the CRA approach

- By starting with concrete experiences, students develop a deeper understanding of abstract mathematical concepts necessary for applications to real-life problems, other subjects, and higher-order thinking.
- Neural pathways and learning are optimised when multidimensional approaches are used.
- The use of manipulatives and visual representations makes learning more engaging and enjoyable for students.
- It caters to the students who learn better with visual, kinaesthetic, auditory or reading and writing modes.

Bouck, E. C. et al. (2020) found a large average effect size (g = 0.93) for use of the CRA approach for change-making problems for students with disabilities.

Stroizer, S. et al. (2015) found a large average effect size (d = 1.07) for the use of the CRA approach for elementary students with maths difficulties.

Agarwal, J. & Morin, L. L. (2016), found a moderate-to-large average effect size (g = 0.82) across multiple maths domains (e.g. fractions, algebra) for use of CRA approach

Doabler, C. T. et al. (2016) found a large average effect size (d = 0.76) for Tier 2 maths interventions incorporating the CRA approach.

Hattie's effect size for adjunct aids was 0.64 with a robustness of 4.

Manipulatives and visuals that may be used

Because of the conditions in classrooms and the paucity of resources in India, we suggest the use of the following manipulatives and visual resources until more resources can be provided.

NCERT and many NGOs have created low-cost manipulatives that the schools can buy. If the schools do not have the resources to buy them, teachers/students can use the resources given below:

Manipulatives

Use of toothpicks and rubber bands: These can be used for understanding the concept of numbers and operations on whole numbers, geometry, and problem-solving. Students can be asked to buy them.

How to use them for numbers and operations on numbers in primary school: see my article or books available free on Google Play.

https://www.researchgate.net/publication/343945617_Use_of_Sticks_as_an_Aid_to_Learning_of_mathematics_for_classes_I-VIII

https://play.google.com/store/books/details?id=_1w3EAAAQBAJ

https://play.google.com/store/books/details?id=nPI4EAAAQBAJ

https://play.google.com/store/books/details?id=PQQ7EAAAQBAJ

https://play.google.com/store/books/details?id=0uY-EAAAQBAJ

https://play.google.com/store/books/details?id=Ebs0EAAAQBAJ

Tangrams: These can be made by drawing and colouring the figure given below and pasting it on thin cardboard and cutting the pieces. Small children may take the help of parents.

Tangrams can be used for enhancing understanding of concepts of shape, symmetry, transformations, tiling, perimeter, area, fractions, mathematical reasoning, communication, and spatial visualisation.

The use of tangrams to make different objects, people, and animals can also be integrated with art.

For some activities that may be used with tangram pieces to enhance maths learning, see my article:
https://www.researchgate.net/publication/343889847Activities with Tangram_for_Learning_mathematics

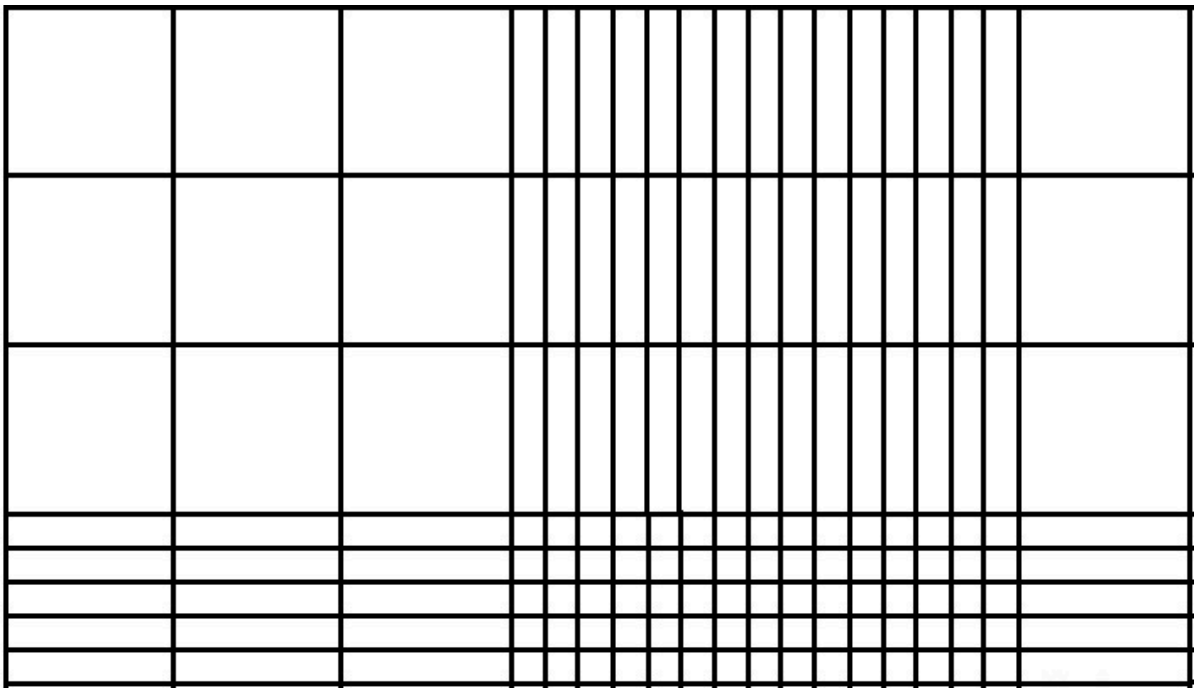
Paper cutting and folding: It can be used for enhancing communication, understanding of concepts, and reasoning in geometry and fractions. The

activities also provide opportunities for the exploration of ideas and verification of results.

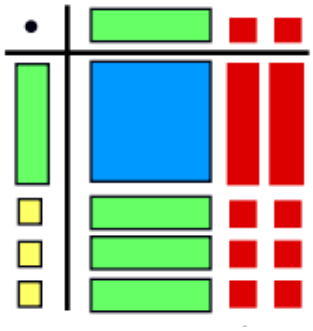
It can also encourage creativity by integrating it with the art of origami.

For some activities that may be used with paper cutting and folding, see my article:
https://www.researchgate.net/publication/343945623_Cutting_and_Folding_Activities_with_paper_classes_I-_VIII

Algebra tiles: These can also be made by making 2 copies of the sheet given below and asking students to colour large squares blue, rectangles green and small squares yellow to represent $x^2,\ x\ and\ 1$ respectively on one sheet and rectangles red and small squares red to represent $-\ x\ and\ -\ 1.$ Paste the sheets on thin cardboard, cut them, and save them in a box.



For example, using tiles, $(x\ +\ 3)(x\ -\ 2)\ =\ x^2\ +\ x\ -\ 6$ can be represented by



Algebra tiles can be used to teach operations on integers, simplifying algebraic expressions, solving linear equations, solving word problems, modelling, adding, subtracting, multiplying, and factoring polynomials.

For how to use algebra tiles, see Using Algebra Tiles Effectively

Visual Representations

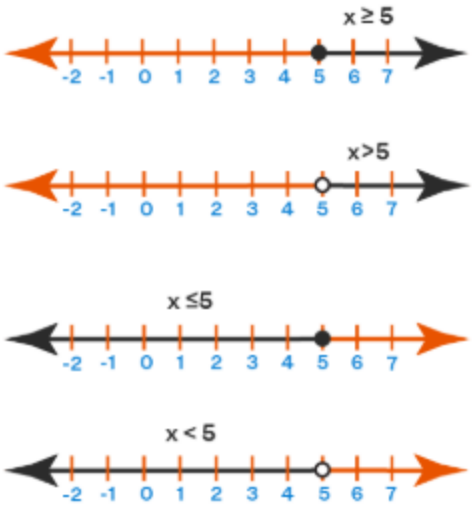
Pictures of objects, number lines, a table of 1-100, and addition and multiplication tables can be used for learning numbers and operations on them. Textbooks can provide them, or students can use graph paper to make these tables.

Number lines: Number lines are straight lines with numbers evenly spaced in order along the line. They are usually horizontal but can have vertical representations too, e.g., in thermometers.

Number lines can be used for

- Representation of whole numbers, integers, decimals, fractions and rational and real numbers by labelling appropriate values on the equal markings on a number line.
- Comparing numbers: The numbers on the right of a number are larger than the number, and the numbers on the left of a number are smaller than the number
- Addition, subtraction, multiplication and division of whole numbers and addition and subtraction of fractions with the same denominators.

- Estimation and rounding of numbers.
- Graphing inequalities on a number line.



Note: A filled circle on the particular number indicates that the number is included, and an empty circle on a number indicates that the number is not included. The arrows on the end denote that the line extends from $-\ \infty\ to\ +\ \infty$.

Dot Paper: Dot paper that can be used for enrichment at levels 1 and 2 of Heiles that would help in preparing students for high school geometry.

Heiles, G. & Heiles, P. (1957) translated in English by Crowley, M. L. (1987) proposed that there are 5 levels of thought at which we can discuss, learn and teach geometry. At level 0, the students perceive figures by their appearance as a whole. When they call a figure a square, they react to the total figure and not to its specific properties, such as all its angles being right angles and all its sides having equal lengths. At level 1, students can analyse figures and see such properties. They are also able to identify figures by their properties, e.g. a figure that has all its sides of equal lengths and all its angles as right angles is a square. They still do not see relations between figures; e.g., a square is a rectangle. At level 2, the students begin to see relationships between figures, e.g. a rectangle is a parallelogram. At level 3, they can follow the proof of theorems, and they can develop a sequence of statements to deduce one statement from

another. Level 4 is the most rigorous level at which students have a deeper understanding of mathematical structures and can justify statements made about structures.

For how to use dot paper for enhancing learning, see my article: https://www.researchgate.net/publication/343945301_Use_of_Dot_Paper_as_an_Aid_to_Enhance_Learning_of_mathematics_for_Classes_I_-_VIII

Graph papers: Square, circular, and isometric graph paper can be used to enhance the understanding of the concepts of numbers, operations on numbers, geometry, measurement, presentation of data, and linear and quadratic functions. These help to enhance communication, problem-solving, and further learning.

For how to use graph paper for enhancing learning, see my article: https://www.researchgate.net/publication/343945490_Activities_with_graph_paper_for_enhancing_the_learning_of_mathematics_for_classes_V-VIII

Graphs: Graphs are a fundamental and versatile tool in mathematics, used to visualise relationships and model real-world problems and can be used from elementary school maths to research in math. They are used in secondary school for

1. Visualisation and understanding of functions.

   It shows its roots (x-intercepts), y-intercepts, slope, maxima, minima, and intervals of increase/decrease of the function and its shape: for example, a straight line (linear), a parabola (quadratic), a wave (trigonometric), or a curve (exponential/logarithmic)?

2. Modelling relationships and solving equations and inequalities.

   Graphs provide solutions to equations and systems of equations.

3. Data analysis and statistics

Graphs aid in identifying patterns, trends, and distributions in datasets, e.g. histograms, bar charts, and line charts; proportions of a whole, e.g. pie charts; and relationships between variables, e.g. scatter plots.

Table of 1-100: It can be used for learning

- Counting, number recognition, connecting number names to numerals. recognising the order of numbers and comparing numbers.
- Learning about even, odd, and prime numbers.
- By moving across the chart, students can add or subtract numbers.
- Skip counting, which aids in learning multiplication facts.

Addition Table: Students can use it for finding addition and subtraction facts and relations between them, seeing patterns, and deriving the commutative law of addition.

Multiplication Table: Students can use it for finding multiplication and division facts, the squares of numbers, seeing patterns, and deriving the commutative and distributive laws of multiplication.

Reading and creating tables: Tables are a fundamental and highly versatile tool in mathematics; they provide an organised, structured way to display data. The data may be the result of an experiment or survey or related to a household budget or bank statement. The tables can reveal patterns, relationships, and trends that might be difficult to see in a block of text or a series of equations. They also support problem-solving by organising data.

2. Connect concepts to concepts learned earlier

For example,

- Addition and subtraction are inverse operations; multiplication is repeated addition, and division is repeated subtraction and is the inverse of multiplication.

- The relationship between decimals, fractions and percents
- Similarities and differences in different types of special quadrilaterals.

For more on connecting concepts with concepts learnt earlier, see Nolan E. C. et al. (2016) and Dixon, J. K. et al. (2016).

3. Applications of maths concepts to real-world scenarios

Tamur, M. et al. (2021) conducted a meta-analysis of contextual teaching-learning of 21 primary studies from 2010 to 2020 that involved 1349 students. The overall effect size was 0.868, with a standard error of 0.16.

Key Benefits of learning applications to real life

- Students are more likely to be engaged in maths when they see its relevance to their lives.
- Seeing the value of maths in their own lives can motivate students to learn and improve their maths skills.
- By applying maths to real-world problems, students gain a deeper understanding of the concepts, as it activates more regions in the brain.
- It improves students' communication skills.
- Solving real problems encourages students to think critically and creatively to solve problems.

Teaching strategies for connecting maths concepts to real-world scenarios

Ask students to

- Suggest where you find numbers or shapes in your surroundings.
- Suggest situations where percentages, ratios and proportions are used in real life.

- Give examples of variables in real life that vary directly and inversely.
- Solve word problems that require the application of an operation or a combination of operations on numbers for real-world situations.
- Solve word problems involving variables that vary directly and inversely in real life, geography, physics and other subjects.
- Write stories for which a given computation involving an operation or combination of operations will be appropriate.
- Write stories for which you will use proportions.
- Create a budget for a school trip or personal expenses.
- Design and make a scale model of a building.
- Design a garden.
- Collect and analyse data on topics of interest to them, e.g., sports statistics and their favourite type of TV programme.
- Solve word problems using algebra.

Important considerations in teaching applications

1. Ensure that the real-world contexts are authentic and meaningful to students. For some examples see Leinwand, S. (2009) and Libby, J. (2017).
2. Ensure students have access to the resources and support they need to participate in contextual learning activities.
3. Use appropriate assessment methods to evaluate student learning, projects, and presentations.
4. Take students on field trips to places where they can observe maths in action, such as farms, construction sites, banks, or businesses.

5. Invite persons from various occupations to share how they use maths in their work.

6. Applications of maths concepts to other subjects

Ask students to

1. Make a person, animal, or some object with tangram pieces or tessellate by making tiles of paper using Escher's method to connect it to art, 122.
2. Apply scaling to maps in geography.
3. Apply direct and inverse proportions to problems in physics.
4. Apply functions to economics, physics and other subjects

2. Knowledge of Facts, Vocabulary, and Fluency in Procedures

Fluency does not mean getting the correct answer quickly. Russel, S.J. (2000) gives three ideas for mental maths and computational fluency: accuracy, efficiency, and flexibility.

Accuracy includes careful recording, double-checking results and knowledge of number facts and relationships.

Efficiency using a strategy that a student can easily carry out. This may differ for different students.

Fluency is to compose and decompose the numbers, shapes, and expressions in various ways and search for patterns and relationships. Notice the structure of numbers in the base-ten number system, the relationship of the operations to each other, the use of the concept of place value in different operations, and the relationship between other mathematical concepts.

Most of the textbooks have questions on these procedures with increasing levels of difficulty.

For fluency questions in which numbers, shapes, and expressions can be composed and decomposed in many ways to make the solution easy may be added, e.g., find mentally 16 × 7, find the area of figures which can be decomposed into figures for which formulae to find the areas are known, and the why of different steps in algorithms.

Effective teaching strategies for knowledge of facts, vocabulary, and fluency in procedures

Use **explicit teaching** for these**.**

Explicit Teaching (Mathematics):

An instructional approach in which the teacher clearly states the learning objective, explicitly explains and models a mathematical concept or procedure step-by-step, and makes their reasoning visible through worked examples and think-alouds. It follows a structured sequence—*I do* (teacher modelling), *we do* (guided practice), and *you do* (independent practice)—with frequent checking for understanding and immediate feedback.

Explicit teaching reduces cognitive load, prevents misconceptions, and supports mastery by providing clear, direct instruction before gradually releasing responsibility to the learner.

Stockard, J. et al. (2018), in a meta-analysis of half a century of research, found explicit teaching approaches had an average effect size d = 0.54 on maths achievement.

The effects were consistent across grade levels (K–12) and student populations (e.g. general education, at-risk learners).

Gersten, R. et al. (2009) found an average d = 0.72 for explicit instruction components (e.g. step-by-step modelling, guided practice) for students with maths difficulties. Most effective strategies included visual representations (e.g. diagrams, number lines) and think-alouds during problem-solving.

Hattie, J. (2017), found explicit teaching methods (e.g. direct instruction, worked examples) had an average effect size d = 0.59 on maths outcomes.

Teacher clarity (explicit explanations, objectives) was among the top influences (d = 0.75).

Kroesbergen, E.H. & Van Luit, J. E. H. (2003) found explicit instruction was significantly more effective (ES = 0.58) than constructivist methods (e.g. discovery learning) for students with maths difficulties.

Baker, S. et al. (2002) found explicit instruction in early numeracy (e.g. counting strategies, arithmetic) had an d = 0.50 for low-achieving students. Immediate feedback and corrective modelling were critical components.

Hughes, E.M. et al. (2016) found explicitly teaching maths vocabulary and problem-solving steps improved outcomes (d = 0.65) for diverse learners.

Hattie's effect size for explicit teaching was 0.63 with a robustness of 5.

We recommend explicit teaching for knowledge of facts and maths vocabulary, carrying out the procedures, and knowledge of problem-solving strategies. It is especially efficient for introducing new, complex procedures (e.g. long division, solving equations), remediation for skill gaps, and for students with learning difficulties.

Tips for explicit teaching

- Begin a lesson with a short review or test of prerequisites.
- Explicitly state what students will learn.
- For vocabulary and definitions, provide for retrieval after learning them.
- Use fact families to develop the automaticity of number facts, e.g.

  $3 + 5 = 8, 5 + 3 = 8, 8 - 3 = 5, 8 - 5 = 3;$

  $4 \times 6 = 24, 6 \times 4 = 24, 24 \div 6 = 4, 24 \div 4 = 6.$

- Think aloud and model steps for procedures and problem-solving strategies.
- Provide guided practice until students can carry out the procedure correctly.
- Provide independent practice for fluency in procedures.

  The effect size of practice tests in Hattie's study was 0.49 with a robustness of 4.

- Provide **scaffolding** for complex procedures.

  Scaffolding refers to a teaching strategy that provides temporary support to students until they can master a task. It involves breaking down learning into manageable chunks and providing tools, resources, or guidance at each step. As students gain confidence and skills, these supports are gradually removed, allowing them to do the task independently.

  Hattie's effect size for scaffolding and situated learning was 0.52 with a robustness of 4.

- Check regularly for understanding so that misconceptions can be addressed immediately.
- Ask students to verify results using a different procedure, e.g. subtraction using addition or division using the relation dividend = quotient × divisor + remainder) and factorisation in algebra by multiplying the factors.
- Provide practice in rounding numbers and their use in estimating and judging the reasonableness of answers for computations.
- Provide spaced practice and interleaving with related topics for retention and consolidation of knowledge of facts, vocabulary and fluency in procedures.

- For developing fluency in procedures, ask questions in which can be solved in many ways or more easily the use of algorithms and discuss them, e.g., find mentally 16 × 7, find area of figures which can decomposed into figures for which formulae to find the areas are known, use formulae to simplify computations, e.g. find 47 × 33 ((40+7)×(40-7)=1600-49=1551)

3. Development of Communication Skills in Mathematics

Communication is an essential component of mathematical understanding. For students to understand problems and communicate solutions effectively, they should be able to

- Explain maths vocabulary.
- Understand word problems.
- Describe the steps in solving a problem.
- Present solutions to problems in class.
- Write mathematical proofs.
- Use diagrams, graphs, charts, and models to illustrate mathematical concepts.
- Convert a representation of a concept into another representation, e.g. a table of inputs and outputs to an equation or graph.
- Ask clarifying questions from teachers or peers and participate in class discussions.
- Do peer tutoring.
- Demonstrate their metacognitive thinking in homework, class discussions or a journal.

Smith, J. R., & Johnson, L. M. (2018), in a meta-analysis of 42 studies with an N = 67,890, found a moderate correlation (r = 0.32, 95% CI [0.28,

0.36]) between communication competencies (e.g. verbal ability, clarity of expression) and overall academic achievement, including mathematics performance.

Chen, H., & Baker, S. K. (2020), in a meta-analysis of 28 intervention studies with an N = 15,200, revealed that structured communication activities (e.g. peer discussions, explanatory questioning) in maths classrooms significantly improved students' problem-solving performance (g = 0.45, 95% CI [0.39, 0.51]). The effect was stronger in middle school compared to elementary settings.

For the effect size of various communicating strategies on achievement in Hattie's study, see Table 2.6 in Chapter 2.

Benefits of good communication skills

Students with good communication skills are more likely to

- Develop a deeper understanding of concepts and identify and correct errors in their reasoning.
- Understand maths problems given in words.
- Participate in discussions of mathematical problems that help students develop critical thinking and problem-solving skills.
- Identify and address weaknesses in their problem-solving strategies through feedback from peers and teachers.
- Articulate their mathematical ideas to develop reasoning skills and become more proficient at justifying their solutions.
- Do a self-assessment.
- Do peer tutoring – a very effective learning strategy.
- Do metacognitive thinking and write or express it verbally.

How teachers can foster communication skills

Teachers can use discussion and journaling for the development of communication skills.

Discussion

Discussion in the classroom requires students to actively engage in a verbal exchange about mathematical concepts, strategies, and reasoning, allowing them to share their thinking, justify their answers, and collaboratively build understanding through open dialogue with the teacher and peers.

The strategy for discussion involves the following steps:

- The teacher presents the problem to the whole class or a small group. Computation problems are always presented horizontally (e.g., find mentally 26 + 35) to encourage different mental strategies rather than over-reliance on algorithms.
- Students are given 1–2 minutes to mentally think about the problem and try to find an answer. When students have an answer, they signal quietly to the teacher (e.g., with a thumb against their chest).
- A few students or groups volunteer to share their answers, and the teacher records them on the blackboard without judgement. The teacher then takes answers that are different from those that have been shared.
- Students share how they got their answers with a partner or larger group. Any student can explain any answer on the board. The teacher records the student's name and how he or the group arrived at the answer using words, numbers, and symbols.

The teacher or students then may ask questions for different problems, such as

- Can you explain your strategy for solving this problem?
- Does your answer make sense?

- How do you know this answer is correct?
- Do you all agree with this solution? Why or why not?"
- Can anyone offer an alternative way to approach this question?
- Would someone like to add on?
- How is Ranjana's approach different/similar to Anil's approach?
- Would your generalisation always work? If not, can you give a counterexample?

Klibanoff, R. S. et al. (2019), in a meta-analysis of 36 studies with an N = 22,417, found that high-quality teacher maths talk (e.g. use of mathematical explanations, questioning, and academic vocabulary) had a significant positive effect on students' maths achievement (g = 0.42, 95% CI [0.36, 0.48]). Effects were larger in early childhood (g = 0.51) than in later classes (g = 0.35).

Webb, N. M. et al. (2021), in a meta-analysis of 25 experimental and correlational studies with an N = 8,732, found that structured student maths talk (e.g. peer explanations, think-aloud strategies, and argumentation) improved maths problem-solving skills (d = 0.56, 95% CI [0.48, 0.64]). The effect was stronger in collaborative learning settings than in individual verbalisation tasks.

The effect size of discussion in Hattie's study was 0.82 with a robustness of 1. (It was based on one meta-analysis.)

How to conduct an effective discussion

- Create a supportive classroom culture which encourages students to share their ideas without fear of judgement or ridicule.
- Teach students how to listen attentively to others and ask clarifying questions.

- Ask open-ended questions and follow them with a discussion. For more on discussion, see Parrish, S., 2018; Humphreys, C. & Parker, R., 2015; and Margaret, S. & Stein, B. K., 2020.
- Implement collaborative learning activities to provide opportunities for students to discuss their ideas, debate different approaches, and learn from each other by think-pair-share or solving rich questions in small groups.
- Ask groups to present solutions to rich questions in class and follow it with discussion.
- Provide specific and helpful feedback on students' communication skills, focusing on both strengths and areas for improvement.
- Help students to refine their explanations by asking questions such as "Can you explain that in a different way?" or "Why do you say that?"
- Encourage peer tutoring.

For examples of questions for discussion, see Chapter 6

Journaling

Journaling in mathematics is a reflective practice where students (or researchers) write about their thought processes, problem-solving strategies, observations, and insights related to mathematical concepts.

For journaling, teachers should model it and give prompts for journal entries that may require

Reflection

- What did you learn today?
- What part of today's lesson was most challenging for you?
- What is one question you still have about today’s topic?

- How is it related to some earlier topic?
- How you may use it in the future?

Explanation of concepts

- Explain in your own words what a concept means and how it can be used in real life.
- How would you teach the concept to someone who does not yet know it?
- Represent the concept in many ways?

Making Connections

- How does this concept connect to other concepts you learnt earlier?
- How is it similar to or different from them?
- How it may be used in real-life situations or other subjects?

Writing solutions to rich problems and their extensions

- Give steps for solving the problem.
- How do you know your answer is reasonable or correct?
- What did you learn from discussions in your group or presentations of solutions by other groups?
- After discussion of other strategies in class, would you rather use another strategy for solving such problems in the future?
- How can this problem be further extended or explored?
- If you struggled with some steps, what did you learn from it?

For examples of questions for journaling, see Chapter 6

4. Development of higher-order thinking

Higher-order thinking in maths encompasses number sense; knowledge of facts and maths vocabulary; fluency in mathematical procedures; a deeper understanding of mathematical concepts; manipulation and interpretation of numerical data (e.g. statistics, probabilities); a sense of cause and effect; number sense; spatial reasoning; knowledge of problem-solving strategies; explaining and justifying thinking by using direct proof, contradiction, induction, or counterexamples; and having traits in the affective domain that promote learning.

We also need to provide opportunities for exploring, questioning, working systematically, visualising, conjecturing, proving or disproving the conjectures.

We will mainly focus on approaches that schools with few resources may follow to enhance higher-level thinking.

We assume most of the students do not have access to technology except perhaps a smartphone at home, and teachers have access to at least a smartphone.

Development of number sense

Developing number sense: Number sense provides a foundation for maths. It refers to a person's intuitive understanding of numbers, ability to subitise (recognising numbers presented visually without counting), seeing relationships between numbers, recognising when we can use ish numbers, flexibility with numbers, and ability to decompose, recompose, and manipulate numbers in various ways, making reasonable guesses about quantities or calculations, and estimating and judging the reasonableness of answers by estimation. Estimation involves finding an approximate value of a quantity, calculation, or measurement when an exact value is either unnecessary or difficult to determine. It helps simplify complex problems and make quick, reasonable judgements in real-world scenarios.

It may involve guessing the number of objects in a group, estimating a number or computations involving different operations in maths by rounding, length, area, volume, weight or volume without tools, or use of sample data to predict population parameters, e.g., proportion or mean.

It helps to judge computations for reasonableness, is useful in real-life situations (e.g. budgeting, construction), and is essential in statistics for making predictions.

Teachers can nurture number sense by encouraging mental math, estimation, pattern recognition, and multiple solution strategies, rather than focusing solely on rote procedures.

Recognition of patterns

Maths is a study of patterns. Recognising these helps in solving problems, predicting future events, finding general rules and formulae and deep connections between seemingly different areas of mathematics and the natural world.

It may be developed by providing opportunities for its development over time by asking students to recognise the rules in a series of numbers or shapes, apply it to find the other terms in a series, generalise the rule, and find its use in nature, e.g., in the Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34…, the rule is $F = 0,\ F_1 = 1.\ F_n = F_{n-1} + F_{n-2}$ for n > 2. This pattern appears surprisingly often in nature (e.g., arrangement of leaves, spirals in shells, sunflower seeds).

Development of mathematical intuition

Provide opportunities for its development over time by asking students to

- Play with numbers and shapes.
- Solve puzzles.
- Look for patterns.

- Create mental images of mathematical ideas.
- Recognise the relationship between different mathematical concepts.
- Represent concepts in many ways.
- Reflect on solutions by thinking why a particular method worked or didn't work.
- Explore dynamic visualisations using online apps, e.g., GeoGebra and Desmos.
- Give situations in which **ish** numbers are used.

Development of divergent thinking

Ask students to solve problems with many correct answers, e.g., open middle questions (https://www.openmiddle.com/), or questions with many strategies to solve them. e.g. calculate mentally 16×6 and explain how you calculated it.

Development of inductive reasoning

It involves generalising from several examples of a law, e.g., deriving the commutative and associative laws of addition and multiplication from several examples of those or addition and multiplication tables.

The effect size of inductive teaching in Hattie's study was 0.60 with a robustness of 3.

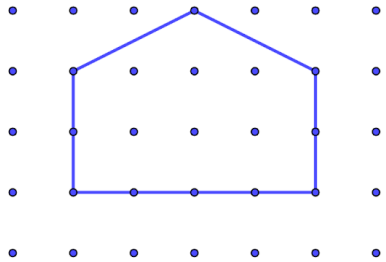
Development of deductive reasoning

It may be developed by asking students to apply general rules to specific cases, e.g.,

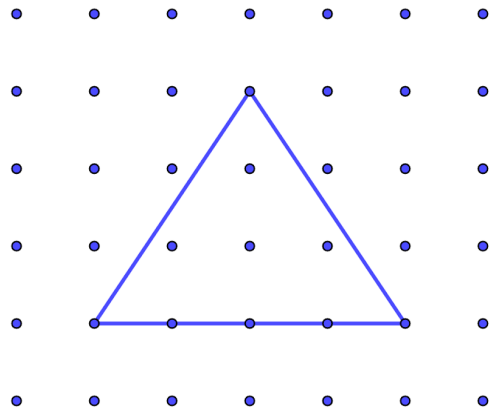
- Pick's Theorem: The area of a polygon on dot paper, when its vertices are on dots, is given by $A = I + \frac{b}{2} - 1$, where I is the number of grid

points inside the polygon and b is the number of grid points on the sides of the polygon. For example, the area of the polygon given below is

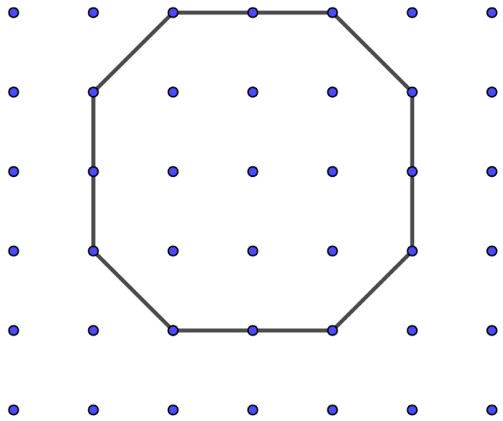
$6 + \frac{10}{2} - 1 = 10$ unit².



Find the area of the polygons given below:



(i)



(ii)

- Give proofs of theorems by using postulates and other theorems.

Development of spatial reasoning

Spatial reasoning refers to the ability to visualise objects and their movements in the mind, understand relationships between objects (e.g. distance, direction, and orientation), manipulate objects mentally or physically (e.g. rotating, flipping, or transforming shapes), interpret and create visual representations (e.g. maps, graphs, and diagrams) and perceive how objects would look when viewed from different sides.

It may be developed by asking students to

- Create shapes with tangrams.
- Create shapes congruent to a shape on a geoboard or dot paper.
- Identify if two shapes are congruent.
- Identify and create shapes that will tessellate using Escher's method.
- Transform shapes by translation, reflection and rotation.
- Identify if a shape is the translation, reflection, or rotation of another given shape.

- Identify if a shape is an enlargement or reduction of a given shape.
- Create shapes that are enlargements or reductions of a given shape.
- How objects would look when viewed from different sides.
- Identify and draw 3D shapes.
- Create and identify nets of 3D shapes.
- Identify the shape of intersection when a plane intersects a 3D shape in a horizontal, vertical, or slanting manner.
- Make and interpret maps.
- Build models with given dimensions.

Problem solving

According to Polya G. (1945), problem solving involves

- **Understanding the problem**: It involves being able to
    - Read the problem carefully and identify the given information and the unknown.
    - Restate the problem in your own words to ensure understanding.
    - Draw diagrams or charts to visualise the problem.
- **Developing a plan**: It involves being able to
    - Choose a strategy or method that seems appropriate for the problem.
    - Consider using the visual representation, formula, equation, or algorithm.
    - Break down the problem into smaller, more manageable steps.
- **Carrying out the plan**: It involves being able to

    - Follow your plan and perform the necessary calculations or operations.
    - Check your work at each step to avoid errors.
- **Looking back**: It means reflecting, which involves being able to
    - Review the solution to ensure it is correct and reasonable.
    - Reflect on the problem-solving process to learn from it for future problems.

Specific problem-solving techniques: There are many strategies for solving problems. For example,

- Use hands-on objects or role-play the situation
- Guess and check
- Use logical reasoning
- Make a table
- Make an organised list
- Simplify the problem
- Draw or make a diagram
- Work backwards
- Look for a pattern

Examples of these are given in Chapter 6.

We suggest explicit teaching of problem-solving, as there is a lot of research evidence that explicit teaching is more effective than the discovery approach for solving problems.

Jitendra, A. K. et al. (2015) found schema-based instruction (SBI) teaching students to categorise word problems by problem type had a large effect (g = 0.85) on solving maths word problems.

Explicit strategy instruction was particularly beneficial for struggling students.

Kroesbergen, E. H. & Van Luit, J. E. H. (2003) found explicit instruction (direct teaching of problem-solving steps) was more effective (d = 0.58) than constructivist approaches for students with maths difficulties. Strategy instruction improved both computation and problem-solving skills.

Gersten, R. et al. (2009) found explicit strategy instruction (e.g. step-by-step problem-solving methods) had a strong effect (d = 0.72) on maths performance. The most effective components included visual representations (e.g. diagrams, graphic organisers) and gradual release of responsibility (I do → We do → You do).

Fuchs, L.S. et al. (2008) found explicit teaching of problem-solving strategies and self-regulation (goal-setting, self-monitoring) led to greater gains (d = 0.62) than traditional instruction. Students maintained improvements over time, showing long-term transfer effects.

Powell, S.R. et al. (2021), in a meta-analysis, found schema-based instruction significantly improved word-problem solving (d = 0.82) across multiple studies. Explicitly teaching problem schemas (e.g. "total vs. difference problems") helped students generalise strategies to new problems.

Hattie, J. (2009) effect size was d = 0.61. The most effective strategies included worked examples, guided practice, and metacognitive questioning.

How Teachers can enhance higher-order thinking Teachers can enhance higher-order thinking by providing opportunities for solving rich tasks which require many of the skills and abilities discussed earlier.

Rich tasks

A rich task is a task for which a learner can't see an immediate known solution path. It may be a puzzle, a game, a maths problem, an experiment, a project or an enquiry task. It presents a high level of cognitive demand and requires an understanding of concepts, knowledge of facts, fluency of procedures, and strategies for solving problems, as well as spatial, inductive, deductive, and spatial reasoning, judging the reasonableness of results, and justifying and communicating the solution.

Characteristics of rich tasks

- It is engaging.
- It focuses on learning goals and requires substantive maths to reach a solution.
- It requires multiple steps and incorporates many mathematical skills and abilities.
- It allows for multiple approaches and solutions to foster creativity and encourage flexible thinking.
- It may have many correct answers.
- It has a low floor and high ceiling so that students with lower levels of proficiency are engaged, and advanced students can explore extensions or variations.
- It relates to contexts that interest students.
- It provides opportunities for students to discuss mathematical ideas with peers to clarify their thinking and learn from them by working collaboratively.
- It requires students to analyse, evaluate, and justify their reasoning to encourage a deeper understanding of mathematical concepts and develop critical thinking skills.

What is considered a rich task varies with the student's prior knowledge and experiences.

Kyriakides, L. et al. (2019), in a meta-analysis, found higher-order questioning (e.g. "Why does this method work?") had a moderate effect (d = 0.48) on maths achievement. Teachers who used rich questions promoted deeper conceptual understanding over rote memorisation.

Rittle-Johnson, B. et al. (2017) found dialogic questioning (e.g. "Can you explain your strategy?") improved problem-solving accuracy (d = 0.42). Effects were stronger when teachers followed up with probing questions to refine student thinking.

Hattie, J. (2017), found higher-order questions (e.g. analysis, evaluation) had a d = 0.46 in maths.

The impact doubled when paired with wait time (more than 3 seconds for student responses).

Chin, C. (2006), found metacognitive questions (e.g. "How did you decide to use this approach?") boosted maths reasoning skills (d = 0.38). Students in classrooms with rich questioning showed greater transfer of skills to novel problems.

Weber, K.E. et al. (2020) found Socratic questioning (e.g. "What assumptions are you making?") increased conceptual maths performance (d = 0.55). It was most effective when integrated with collaborative problem-solving.

Teaching strategies for solving rich problems

While rich tasks promote higher-order thinking, teaching and tackling such tasks require more effort and deeper thinking on the part of teachers and students.

For rich questions

- **Use collaborative settings for rich tasks.**

  Collaborative learning involves students working together in pairs or groups to achieve a common goal. It's a method that has shown significant benefits in education and beyond.

  Kyndt, E. et al. (2013), in a meta-analysis of 56 studies with an N = 34,511, found that collaborative learning strategies (e.g. peer tutoring, group problem-solving) significantly improved maths performance compared to traditional instruction (g = 0.52, 95% CI [0.45, 0.59]).

  Effects were stronger for students in secondary than in primary schools, and pairs (d = 0.55) outperformed larger groups (d = 0.42).

  Slavin, R. E., & Lake, C. (2008), in a meta-analysis of 44 experimental studies with an N = 12,800, found that structured small-group collaboration led to greater maths gains (d = 0.61, 95% CI [0.53, 0.69]) than individual learning.

  Benefits were highest when groups received explicit roles and accountability measures.

  Hattie's effect size for collaborative learning was 0.34.

  Advantages of collaborative learning

  - By working together, students can tackle complex problems from multiple perspectives, leading to more creative and effective solutions.
  - Observing peers solve problems in different ways is a powerful learning tool.
  - Collaborative learning fosters effective communication and teamwork skills.
  - Explaining concepts to peers reinforces understanding and helps identify knowledge gaps.

- Working in groups can make learning more enjoyable and engaging, leading to increased motivation and participation.
- Many real-world jobs require collaboration and teamwork. Solving problems in groups will prepare for such careers.
- Successfully working in a group can boost students' self-confidence and belief in their abilities.

- **Provide a supportive environment**

  It involves creating a classroom climate where students are not afraid of being wrong, receive constructive feedback, and experience positive teacher–student and peer relationships.

  Such an environment promotes confidence, persistence, and engagement, helping learners develop a growth mindset and deeper understanding of mathematical concepts.

  Encourage development of a growth mindset.

  - Reinforce the idea that persistence and efforts are keys to success, whatever the innate ability.
  - Tell students struggle is good. As when we struggle, our brain pathways form, connect and strengthen.

    Moser, J. et al. (2007), in a study on the brain's response to mistakes, monitored the brains of students with MRI technology when they took a test. The scans revealed that when students made mistakes, their brains were more active in strengthening and growing neural pathways than students who got correct answers

- **Selection of rich tasks**

  Select tasks for which students have the prerequisites and that lend themselves to multiple approaches and creative solutions and are at the edge of students' ability to foster a deeper understanding of concepts and avoid boredom or frustration for students.

- **Introduce problems clearly**, ensuring all students understand the problem by allowing individual "think time" before students begin working in groups.
- **Formation of groups**

  Form heterogeneous groups by mixing students with varying levels of understanding and strengths to encourage peer teaching and diverse perspectives.
- **Tell students how to work in groups**
  - Listen attentively, disagree respectfully, and have equitable participation.
  - Ask questions to understand the thinking of others.
  - Ask questions that enable the group to go deeper.
  - Discuss different strategies for solving the problem and choose one to solve the problem
  - Organise a presentation so that others can understand the group's thinking.
  - Teachers may assign/suggest roles to structure discussions to ensure all students in a group participate. Boaler, J. (2024) suggests the use of roles such as **includer**: who ensures participation of everyone; **connector:** who reminds the group to look for connections and reasons for each mathematical statement, **synthesiser**: who attends to time, space (for creative thinking) and resources; **questioner**: who asks lot of questions and encourages curiosity in the group; and **spy** who watches what other groups are doing. Change students for these roles for different problems.
- **Observe interactions and problem-solving processes** of groups by circulating among the groups. Provide targeted support and guidance

without directly giving away the solution. Focus on helping students overcome roadblocks themselves.

- **Discuss solutions of different groups:** Bring the class back together to share their solution strategies and approaches and facilitate a discussion that focuses on the mathematical concepts involved, the reasoning behind different strategies, the connections between them, the strengths and weaknesses of different approaches, and the justification of their solutions by encouraging students to ask questions from the presenting groups.
    - Ask students what worked well, what challenges they faced, and what they learnt about working together.
    - Guide students to identify efficient or elegant solutions.
    - Provide opportunities for students to reflect individually on their learning, their contributions to the group, and any insights they gained through journal entries, exit tickets, or brief written responses.
    - Praise students for their perseverance and problem-solving process, not just the correct answer.
    - Help students develop awareness of their thinking processes by asking them questions like "What strategy did you use?" and "How does it differ from the strategy of another group?"or "How can you be sure your answer is correct?"
    - Promote communication skills by asking students to convince themselves and others of the validity of their work. This enhances deep understanding and confidence in their capacity to learn.
- **Providing opportunities for development of metacognition**

  Metacognition, or "thinking about thinking", is a crucial cognitive skill that enables individuals to monitor and regulate their learning. It

involves understanding one's thought processes, strengths, weaknesses, and strategies for learning. It helps in developing a deep understanding, enhances problem-solving abilities, and develops a love for the subject.

Students who acquire metacognitive skills can

- Effectively monitor their understanding, identify gaps in their knowledge, and adjust their learning strategies accordingly.
- Break down complex problems into smaller, more manageable steps, evaluate different approaches, and persevere through challenges.
- Perform better on problem-solving and complex tasks.
- Develop critical thinking abilities, such as analysing information, evaluating arguments, and making informed judgements.
- Are more engaged in their studies, as they know their learning processes and progress.

The Programme for International Student Assessment (Wang, X.S., Perry, L.B., Malpique, A. et al., 2023) considered the learning approaches of 15 million students (who had taken PISA in 2015) and their relationship to maths achievement. The students with a relational or self-monitoring approach had the highest achievement.

Metacognition was a stronger predictor of maths achievement than general intelligence.

The effect was more pronounced in older students (secondary and higher education) compared to elementary students.

Ohtani, K. & Hisasaka, T. (2018), in a meta-analytic review of the relationship among metacognition, intelligence, and academic performance, found a moderate positive correlation ($r = 0.38$) between metacognition and maths performance.

Schneider, W. & Artelt, C. (2010) found metacognitive strategies (planning, monitoring, evaluation) significantly improved maths performance (d = 0.45).

Training in metacognitive skills had a larger impact on problem-solving tasks than on basic arithmetic.

Dignath, C. & Büttner, G. (2018), in a meta-analysis, found metacognitive interventions (e.g. self-regulation training) had a moderate positive effect on maths performance (g = 0.50).

Explicit instruction in metacognitive strategies was more effective than implicit approaches.

Jacobse, A. E. & Harskamp, E. G. (2012) found metacognitive monitoring accuracy was moderately correlated with maths problem-solving success (r = 0.32).

Students with better metacognitive awareness made fewer errors in complex maths tasks.

Wang, M. C. et al. (2020) found metacognitive strategies were among the top five predictors of maths achievement across studies. The effect size of metacognition on maths performance was comparable to that of prior knowledge and motivation.

Hattie's effect size for students engaging in metacognition was 0.52 with a robustness of 5

Development of metacognition

Teachers can develop metacognition in students by

- Emphasising the value of metacognition.
- Asking them to respond to a checklist suggested by Jo Boaler in https://www.mathsish.org/activities/handout-reflection

- Asking students to keep mathematics journals where they can reflect on their learning, identify areas of strength and weakness, and set goals.
- Encouraging students to analyse their mistakes and learn from them. This can help them develop a deeper understanding of mathematical concepts and improve their problem-solving skills.

Sources of Rich Questions

There are many resources available on the internet:

- https://www.openmiddle.com

  These are problems with a single answer but with many ways to reach the answer. This site gives many examples of such problems organised by topic, grade, and skill level.

- https://nrich.maths.org/

  This is a multifaceted site from the University of Cambridge in Great Britain. It has rich problems and many informative articles and discussions on the site.

- Rich Mathematical Tasks | Virginia Department of Education

  This site includes rich problems for classes K-12

- https://www.youcubed.org

  It provides learning opportunities, videos, lessons, and evidence-based ideas to support maths instruction and learning for classes KG-12.

  https://illustrativemathematics.org/

  It provides a high-quality, freely available mathematics curriculum, professional learning resources, and assessments for classes KG-12. The curriculum is designed to help students develop a deep

understanding of mathematical concepts through problem-solving, reasoning, and modelling and provides professional development opportunities for teachers to learn how to effectively implement the curriculum and access a variety of resources, including lesson plans, assessments, and professional development opportunities.

- Rich Tasks - maths For Love

  Gives games, puzzles, and tasks that engage students and promote the habits of mind mathematicians need: perseverance & pattern-seeking, courage & curiosity, organisation & communication.

- https://us.corwin.com/landing-pages/rich-tasks

  Gives a few rich questions for classes KG-12.

- https://www.aliciaburdess.com/teacher-resources

  Alicia Burdess has a collection of problems for classes 2, 3, and 8 for the book on Thinking Classrooms by Liljedahl, P. (2020) and suggestions for teaching them.

Creating and scoring rich tasks

For creating your own rich questions and scoring such tasks, see

Blank-Rich-Task-Template and Rich Mathematical Task Rubric by Virginia University. Rich Mathematical Tasks | Virginia Department of Education

Modify questions that are normally used for practice and testing into rich tasks
https://docs.google.com/document/d/1_X8yD6gqi9unt8p9INhpg81Cb6BLzX-_pJWdK5yrAEk/edit?tab=t.0

Examples of some rich questions are given in Chapter 6.

Providing for Individual Differences in Mathematics

There are several types of individual differences which exist among students, e.g., general ability, prior knowledge of a topic, a feeling of self-efficacy in math, mindset, working memory, motivation to learn, preferred mode of learning, such as visual, auditory, or kinaesthetic, learning disabilities, language barriers, or cultural differences that may impact math learning.

It is crucial to create an inclusive learning environment that fosters the academic, emotional, and social growth of every student.

How to Identify Individual Differences

Teachers can identify individual differences by observation, interviews, use of records, use of tests, use of teacher-student conferences, interviews with parents, and use of attitude tests.

How to Provide for Individual Differences

The teachers can provide for individual differences by

- Testing for prerequisite knowledge and providing for remediation.
- Use multiple representations of concepts and procedures for a deeper understanding of concepts so that students that are adept at learning by different modes may understand.
- Using low-floor and high-ceiling tasks so that the tasks are accessible to students with low proficiency (low-floor) and provide opportunities for deep and extended thinking (high-ceiling) for advanced students.
- Use think-pair-share or provide opportunities for working in collaborative settings in small groups that enable learning from peers.
- Asking students to present solutions to rich questions that provide

opportunities to see different strategies for solving problems.

- Circulating in the room and providing additional support and guidance to students having difficulty.
- Differentiating homework to provide additional practice to students having difficulty and enrichment activities for advanced students.
- Regularly monitoring student progress through quizzes, observations, and exit tickets to identify individual strengths and weaknesses and adjust instruction accordingly.
- Providing immediate feedback on what students know and what they need to work on.
- Using maths discussions to provide clarifications and deeper learning.
- Allowing students to work at their own pace through differentiated materials and activities (e.g. videos)
- Developing traits in the affective domain that enhance learning.
- Reducing math anxiety.
- Creating a supportive classroom environment where students feel valued and comfortable in taking risks and asking questions.

For more on providing for equity in the classroom, see Chapter 6 in Boaler, J. (2022), and for some national data on inclusiveness at the school level for Children With Special Needs (CWSN), see NCERT (2024).

Gauging Effectiveness of Different Teaching Strategies

The effectiveness of any strategy for teaching and learning also depends on the context. While choosing a new strategy, teachers should also rely on their own experience, knowledge of students, facilities available to them and what

resonates with them. The new strategy should be evaluated after its implementation to gauge its effectiveness.

Key Takeaways

The research evidence suggests the use of the following teaching strategies is effective for different objectives.

Understanding concepts

- Use of multiple representations of concepts, e.g., manipulatives such as bundles of sticks or linking cubes, tangrams, number lines, graph or dot paper, tables of 1-100, addition and multiplication.
- Applications of maths concepts to real-world scenarios
- Applications of maths to other subjects
- Ask students to recognise patterns in a series of numbers given in numerals or shapes, find the next, 10^{th}, n^{th} term, and state the rule for the series.
- Recognise the use of **'ish'** numbers or shapes in real life.
- Use questions that will aid in developing number sense, divergent thinking and deductive, inductive and spatial reasoning.

Knowledge of Facts, Definitions and Fluency in Procedures

- Provide opportunities for retrieval and spaced practice for retention of facts and vocabulary.
- Use fact families for automaticity of number facts.
- Provide opportunities for spaced practice and interleaving for fluency of procedures.

- Provide explicit teaching with guided practice for learning procedures with feedback and then independent practice and interleaving for consolidation of learning.

Development of Communication Skills in Mathematics

- Use classroom discussion for open and rich questions.
- Ask students to write in a journal answers to open and rich questions.
- Provide opportunities for reflection in discussion and journaling.

Problem solving

- Model different problem-solving strategies and provide opportunities for their application.

Development of higher-order thinking

Higher-order thinking in maths encompasses number sense, knowledge of facts and maths vocabulary, fluency in mathematical procedures, a deeper understanding of mathematical concepts, manipulating and interpreting numerical data (e.g. statistics, probabilities), a sense of cause and effect, number sense, spatial reasoning, knowledge of problem-solving strategies, explaining and justifying thinking by using-direct proof, contradiction, induction, or counterexamples, and having traits in the affective domain that promote learning.

We also need to provide opportunities for exploring, questioning, working systematically, visualising, conjecturing, proving or disproving the conjectures, explaining, generalising, and convincing.

Teachers can enhance higher-order thinking by providing opportunities for solving rich tasks, which requires many of the skills and abilities discussed earlier.

A **rich task** is a task for which a learner can't see an immediate known solution path. It may be a puzzle, a game, a maths problem, an experiment, a project or an enquiry task.

Solving rich tasks will be facilitated by

- Use collaborative settings for rich tasks.
- Provide a supportive environment
- Encourage development of a growth mindset – the belief that students' talents and abilities can be developed over time through effort and persistence, whatever the innate ability.
- Encourage students to embrace challenges and learn from mistakes and tell them mistakes are good, as when we make mistakes, our brains grow. It is especially beneficial for at-risk and underprivileged students.
- Work out with students rules for working in groups.
- Introduce problems clearly, ensuring all students understand the problem by allowing individual "think time" before students begin working in groups.
- Choose problems for which students have the prerequisites and that lend to multiple approaches and creative solutions and are at the edge of students' ability to foster a deeper understanding of concepts.
- Use low-floor and high-ceiling tasks to avoid frustration for students who are at the beginning level and boredom for advanced students.
- Discuss solutions of different groups in the class, including reflection on their solution.

6

Assessment and Feedback in Mathematics

Assessment is a systematic process of gathering, analysing, and interpreting evidence to determine students' understanding, skills, and progress in mathematical concepts. Assessment serves multiple purposes in mathematics education

Types of Assessment

1. **Diagnostic Assessmen**t: It identifies students' prior knowledge and misconceptions before instruction e.g. pre-tests. It is crucial for learning.

   Hattie's effect size for prior achievement was 0.73 with a robustness of 5.

   For a diagnostic test and remedial teaching at the beginning of class VI, see my paper: https://www.researchgate.net/publication/343944464_Diagnosis_of_Weaknesses_and_Remedial_Teaching_in_Class_VI

2. **Formative Assessment:** It seeks evidence on the attainment of objectives of a course while the course is in progress. It aims to enhance the attainment of objectives by guiding teaching and learning.

3. **Summative Assessment**: It is concerned with evaluating student learning at the end of an instructional period (e.g. unit tests, midterm, final exams) for grading, promotion, certification and communication with parents.

   For unit tests and annual exams for classes VI-VIII see my monograph: https://www.researchgate.net/publication/343944483_Internal_Evaluation_in_mathematics_for_classes_VI-VIII

4. **Evaluative Assessment**: Measures effectiveness of a curriculum (e.g. National Achievement Surveys).

We will focus only formative assessment for guiding teaching and learning.

Effectiveness of Formative Assessment

Assessment for learning is concerned with seeking evidence on the attainment of objectives of a course while the course is in progress. It aims to enhance the attainment of objectives by guiding teaching and learning.

- Sortwell, A. et al. (2024) did a meta-analysis of 13 studies and found a high effect size for 1, moderate for 10, and low for 2 studies. However, the robustness of the meta-analysis was low.
- Dylan, W. (2007) synthesizing the research evidence on formative evaluation suggested that
    - Formative assessment produces greater increases in student achievement and is cheaper than other efforts to boost achievement, including reducing class sizes and increasing teachers' content knowledge.
    - Formative assessment that occurs within and between instructional units (medium-cycle assessment) as well as within and between lessons (short-cycle assessment) has been shown to improve students' achievement. Formative assessment across marking periods, quarters, semesters, or years (intervals of four weeks to one year) has not been shown to improve students' achievement.
    - In classrooms where medium- and short-cycle formative assessment was used, teachers reported greater professional satisfaction and increased student engagement in learning.
- Kingston, N. & Nash, B. (2011), in a meta-analysis, found a modest positive effect of formative evaluation on maths achievement, with an overall weighted effect size of d = 0.20. The impact was stronger in studies with shorter durations (less than 12 weeks).
- Klute, M. et al. (2017), in a meta-analysis of formative assessment interventions in K-12 settings found an average effect size on maths achievement (d = 0.25).

The effects were stronger in elementary classes than in middle or high school.

- Gersten, R. et al. (2009), in a meta-analysis on maths interventions for students with learning disabilities, found that formative assessment strategies (e.g. progress monitoring, feedback) significantly improved maths outcomes (d = 0.58).
- Burns, M.K. et al. (2010), in a meta-analysis of maths fluency interventions incorporating formative assessment (e.g. frequent progress monitoring), found a large effect size (g = 0.62) on student performance.
- Hattie's effect size for formative assessment was 0.67 with a robustness of 3.

How Formative Assessment Helps Teachers and Students

It helps teachers

- By demanding thinking about objectives, the match between objectives and instructional activities, and the match between instructional activities and assessment.
- Tracking class-wise progress towards learning goals.
- Monitoring individual students' progress towards learning goals, identifying students needing extra support or enrichment, and tailoring the instruction accordingly.
- Providing targeted feedback to students based on their performance.

It helps students by

- Learning the intended outcomes of learning and devising approaches to learning that are effective in their attainment.
- Knowing if they are achieving the intended outcomes.

- Discovering the types of errors they make and misconceptions.
- Motivating them to be regular and keep up, which is especially important for hierarchical subjects like mathematics.

Tools for Conducting Formative Assessment

Teachers can conduct formative assessment by

1. Written assignments

Maths teachers can use diverse written assignments to enhance conceptual understanding and problem-solving, foster mathematical and critical thinking and improve communication skills.

Teachers can use written assignments in class or as homework to enhance students' learning by asking them to do questions requiring

- Retrieval of knowledge of facts and definitions to consolidate them.
- Practice of procedures to provide fluency with them.
- Spaced practice of maths vocabulary and procedures to retain them.
- Interleaving of different topics for better retention and consolidation.
- Solution of word problems and writing of word problems for given computations.
- Synthesis of learning, e.g. how a shopkeeper, a farmer, or a housewife uses maths in daily life.
- Applications of concepts in other subjects.
- Identification and correction of the mistakes in solved examples..
- Solution of problems
- Writing of reports of the rich tasks carried out in groups.

- Writing of mathematical proofs.
- Reflection on learning experiences (metacognition).
- Documentation of experiments/simulations, e.g. report on a probability simulation using dice rolls.

2. Discussion

A discussion in a maths classroom actively engages students in a verbal exchange about mathematical concepts, strategies, and reasoning. It enables them to share their thinking, justify their answers, and collaboratively build understanding through open dialogue with the teacher and peers.

Classroom discussion in mathematics helps in

- Identification and addressing misconceptions through dialogue.
- Fostering deeper conceptual understanding.
- Enhancing the development of multiple problem-solving approaches.
- Development of communication skills and divergent and critical thinking.

Effective questioning techniques for discussion

For a discussion

- Give students ample time to think before responding. This encourages deeper thinking and allows for more thoughtful answers.
- Call on different students to ensure that all students are engaged and have opportunities to participate.
- Ask follow-up questions to probe students' thinking and to push them to think more deeply.

- Give students time to think about a question individually, then discuss their ideas with a partner, and finally share their thoughts with the class to help students learn from each other.

How teachers may evaluate a discussion

After a discussion, a teacher may note, using a checklist, whether the students

- listened actively.
- disagreed respectfully.
- participated
- the quality of the questions and answers of the participants

3. Self-assessment by students

Self-assessment is a powerful tool that empowers students to reflect on their learning, identify strengths and areas for improvement, and take ownership of their learning. Fostering self-assessment skills helps students develop metacognition, critical thinking, and independence.

Self-assessment involves students evaluating their work, understandings, and learning processes. It includes:

- Identifying what they did well and where they struggled.
- Creating actionable steps for improvement.
- Tracking growth over time, e.g. after solving a set of equations, a student reflects on their accuracy, speed, and understanding of the concepts.

Brown, G. T. L., & Harris, L. R. (2013), in a meta-analysis, found self-assessment improves metacognitive awareness and maths problem-solving accuracy. Students who self-assess regularly show greater

gains in procedural maths skills (e.g. algebra, calculus). Effects are stronger when combined with teacher feedback.

Panadero, E., Brown, G. T. L., & Strijbos, J.-W. (2016), in a meta-analysis, found a moderate average effect size of self-assessment on maths performance (d = 0.45). Benefits are highest when students use rubrics or checklists to evaluate their work, and struggling maths learners showed greater improvement than high achievers.

Andrade, H. L. (2019), in a meta-analysis, found that self-assessment enhances conceptual understanding in maths (e.g. geometry proofs, word problems). Students who reflect on mistakes reduce repeated errors in calculations.

Effects depend on training (guidance on how to self-assess accurately).

Ross, J. A., Rolheiser, C., & Hogaboam-Gray, A. (2002), in a meta-analysis, found self-assessment boosts motivation and persistence in problem-solving in maths.

Students who track their progress outperform peers on standardised maths tests.

Younger students (elementary/middle school) benefit more than high schoolers.

Kostons, D., van Gog, T., & Paas, F. (2012), in a meta-analysis, found self-assessment training improves maths self-regulation (e.g. selecting appropriate strategies).

Students who self-assess during practice problems learn faster than those who don't, and the effects are strongest in complex maths tasks (e.g. multi-step equations).

Sanchez, C. E., Atkinson, K. M., Koenka, A. C., Moshontz, H., & Cooper, H. (2017), in a meta-analysis, found self-assessment in maths leads to small but significant gains in test scores (d = 0.29).

Self-assessment works best for formative assessment (e.g. homework, practice tests), and combining self- and peer-assessment maximises accuracy.

Benefits of Self-Assessment

Self-assessment

- Encourages students to take responsibility for their learning.
- Identify their strengths and weaknesses. By regularly reflecting on their learning process, students can identify challenges, successes, and areas for improvement.
- Helps students understand how they learn and solve problems.
- Enhance communication of mathematical ideas and accurate use of mathematical vocabulary.
- Helps students monitor the growth of their understanding of specific concepts and identify areas where they've overcome challenges over time.
- Develops metacognition. By thinking about their thinking processes, students can understand how they learn best, what strategies work for them, and how to approach problems more efficiently.
- Enhances problem-solving abilities essential for success in various aspects of life.

Strategies for implementing self-assessment

Ask students to write in an exit ticket or journal about a lesson. Provide clear criteria, e.g. share rubrics, checklists, or prompts so students know what to assess.

Exit tickets: Exit tickets are quick, informal assessments that gauge how well learners have grasped the lesson and are given after a lesson.

Rubrics: A rubric is a scoring guide that outlines specific criteria and performance levels for evaluating students' mathematical work, making expectations clear and providing a structured way to assess understanding, reasoning, accuracy, and communication.

Journaling: Maths journaling involves students writing about their mathematical thinking, problem-solving processes, understanding of concepts, and reflections on learning. It focuses on expressing reasoning, synthesising ideas, and making connections in mathematics through written communication.

- Students may also be asked to review their journals regularly to track growth in their learning.
- Teachers should check the entries in journals as frequently as they can or ask students to share entries from their journals in class.

4. Peer Assessments

Peer assessment involves students reviewing and assessing their peers' mathematical solutions, proofs, or problem-solving approaches based on predefined criteria.

Studies show peer assessment in maths improves self-regulation and conceptual mastery when combined with teacher guidance (e.g. Nicol et al., 2014).

Double, K. S., McGrane, J. A., & Hopfenbeck, T. N. (2020), in a meta-analysis, found peer assessment has a moderate positive effect (average $g = 0.43$) on academic performance across subjects, including mathematics.

The effect was slightly stronger in STEM subjects than in humanities, suggesting structured peer feedback is particularly beneficial for problem-solving disciplines and the quality of feedback (clear rubrics, training) significantly influenced outcomes.

Li, H., Xiong, Y., Hunter, C. V., Guo, X., & Tywoniw, R. (2020), in a meta-analysis, found peer assessment led to small to moderate improvements in learning outcomes (d = 0.31).

In maths, peer assessment helped students to develop metacognitive skills (e.g. self-monitoring, error detection) more than in other subjects, and the effect was stronger when combined with teacher guidance.

van Zundert, M., Sluijsmans, D., & van Merriënboer, J. (2010), in a meta-analysis, found peer assessment improves higher-order thinking skills, particularly in structured tasks (like maths problem-solving). Students who engaged in peer assessment showed better conceptual understanding of maths than those who did not, and training students in assessment criteria was critical for accuracy.

Topping, K. J. (2017), based on 43 studies, found that peer assessment was effective in 41 of these studies. Some studies also reported gains in metacognition and transferable skills in addition.

Gielen, S. et al. (2010), in a meta-analysis, found the effectiveness of peer feedback depended on scaffolding (e.g. rubrics, examples).

In maths, formative peer feedback (focused on improvement rather than grades) led to the biggest gains, and anonymous peer assessment reduced bias and increased honesty in feedback.

Strijbos, J.W., & Sluijsmans, D. (2010), in a meta-analysis, found peer assessment is most effective when integrated into collaborative learning (e.g. group problem-solving in maths).

In maths students improved in procedural accuracy (e.g. solving equations correctly) when they assessed peers' work, and combining self-assessment with peer assessment amplified benefits.

Benefits of peer assessment

- Promotes discussion and shared learning.

- Students engage with different problem-solving methods.
- Learners develop the ability to analyse and do critical thinking.
- Saves time for the teacher.

Implementation of peer assessments

- Define expectations (e.g. correctness, reasoning, presentation).
- Demonstrate how to assess using sample work.
- Use rubrics or guided questions for consistency. An example of a rubric is given in Table 6.1.
- Have students revise their work based on peer feedback.

Table 6.1:Rubric for peer assessment of a problem

Criteria	Excellent (3)	Satisfactory (2)	Needs improvement (1)
Accuracy	Fully correct	Minor errors	Major errors
Clarity	Clear steps	Some confusion	Unclear reasoning
Justification	Logical proof	Partial reasoning	No justification

Challenges in peer assessment and how to overcome them.

- Subjectivity in maths → Use detailed rubrics focusing on objective criteria.
- Student hesitation → Provide training and emphasise constructive feedback.
- Time-consuming → Start with small, focused tasks (e.g. single problems).

Peer assessment in maths fosters active learning and critical thinking. When implemented thoughtfully, it complements traditional teaching methods and empowers students to engage deeply with mathematical concepts.

5. Observation

Teacher observation and recording of learning in a maths classroom are crucial for effective instruction and student growth. By effectively observing and recording student learning in their maths classrooms, teachers can gain valuable insights into student understanding, tailor instruction to meet individual needs, and help students achieve their mathematical potential.

Bell, C. A., et al. (2012), in a meta-analysis, found structured teacher observation tools (e.g. CLASShttps://tntp.org/tool/classroom-observation-protocols/ , MQI: https://usprogram.gatesfoundation.org/-/media/dataimport/resources/pdffocusing on teacher-student interactions (e.g. questioning techniques, feedback clarity) correlates with higher maths gains.

Kane, T. J., & Staiger, D. O. (2012), in a meta-analysis, found teachers who make frequent, structured observations (e.g. 4 or more times in a year) improve maths scores by 0.12 standard deviations. (An improvement of 0.12 standard deviations in math scores means that, on average, students performed about one-eighth of a standard deviation better — roughly equivalent to 1–2 months of additional learning progress).

Observations are most effective when paired with specific feedback (e.g. asked more open-ended questions during algebra lessons).

Observation is necessary for hands-on learning and is especially useful for assessment of group interactions and social skills for collaborative learning.

Key considerations in observations

- Observe how students approach problems, explain their reasoning, and make connections.
- Make anecdotal and mental notes and quick checks for understanding
- Employ a range of observation techniques to gain a comprehensive understanding of student learning.
- Use observations to inform instructional decisions and adjust teaching strategies as needed.
- Share observations with students in a constructive and supportive manner to promote self-reflection and growth.

Observation Techniques

- **Formal observations** are planned observations used by teachers, peers or external observers using specific tools (e.g. checklists, rubrics, observation protocols).

  These may be recorded to track students' progress and share information with parents.

- **Informal observations** are ongoing, casual observations throughout the lesson. These can include anecdotal notes, mental notes, and quick checks for understanding.

  These may be recorded using brief, informal notes about specific student behaviours, observations, or insights.

Strategies that Contribute to Effective Formative Assessment

- Clarifying and sharing objectives and the criteria for success.

- Creating effective classroom discussion by asking questions and engaging students in activities and tasks that provide evidence of how students are progressing toward learning goals.
- Observing students' individual or group work and interactions by circulating in the room.
- Providing opportunities for peer learning by solving rich questions in groups. This also helps students to see diverse perspectives on how to solve problems and develop social skills for collaborative problem-solving.
- Encouraging students to take ownership of their learning.
- Ensuring all students feel comfortable participating in discussions and activities. Use techniques like wait time, think-pair and share, and discussions of group work in the whole class.
- Use scaffolded sentence starters and exit tickets to encourage participation from all learners.
- Evaluating not just the final answer but how students arrived at it. Did they collaborate effectively? What strategies did they use?
- Using your observations alongside formative assessments to personalise learning by providing targeted support or enrichment activities based on individual student needs and preferred learning approaches.
- Giving informative rather than evaluative feedback.

Experiment with a variety of assessment methods. Don’t feel pressured to implement everything at once. Try out one or two strategies and adapt them to your specific classroom needs. Reflect on the effectiveness of your chosen assessment strategies. Gather student feedback and refine your approach based on it.

Assessment should be a continuous cycle of gathering information, reflection, and improvement.

Challenges in Formative Assessment and How to Overcome Some of Them

Formative assessment comes with its own set of challenges. Here's an overview of these challenges and some strategies to overcome them:

- Time Constraints: Teachers often have limited time during lessons to conduct in-depth formative assessments while covering the required curriculum.

  To overcome time constraints, do formative assessments during class, e.g. for a concept or procedure, tell students to keep 1, 2, or 3 fingers against their chest: 1 for 'I need more coaching', 2 for 'I understand but sometimes make mistakes', and 3 for 'I understand very well', respectively. The teacher can scan the number of students with 1, 2, or 3 fingers. Students showing 3 fingers may be asked to do some coaching or enrichment activities, and the teacher may help other students.

  Use tools like Google Forms for instant scoring and data analysis if technology is available.

  Use peer assessments.

- Students come with varying skill levels, learning preferences, and prior knowledge, making it challenging to design assessments that meet everyone’s needs.

  To overcome this, provide choices such as use of verbal explanations, drawings, or written solutions.

- Some students may feel anxious about being assessed, even formatively skewing the results.

  To overcome this, don’t use formative evaluation for grading.

- Some students may not engage because they are afraid they will make mistakes.

To overcome this, instil in students that mistakes are opportunities to learn and make our brains grow.

- Providing personal, timely, constructive, and actionable feedback to all students consistently can overwhelm the teachers.

  To overcome this, use standard templates for feedback to save time.

- Analysis of data to adjust instruction according to the student's performance is time-consuming.

  To overcome this, create charts, graphs, or spreadsheets to visualise student performance and look for common misconceptions or areas where a student or the entire class is struggling.

- Decoding students' errors and understanding their thinking processes can be difficult, particularly when responses are ambiguous or incomplete.

- Embedding formative assessments seamlessly into daily maths lessons without disrupting the flow of teaching can be challenging.

Feedback and Learning

Providing timely, individual and actionable feedback to students after an assessment is highly valuable for their learning and growth.

Hattie, J., & Timperley, H. (2007), in a meta-analysis, found a large average effect size (d = 0.79) for feedback on maths achievement, but effectiveness depended on the type of feedback.

Task-level feedback (e.g. correcting errors) was useful for procedural skills (e.g., arithmetic); process-level feedback (e.g. "Explain your strategy") boosts conceptual understanding (e.g. algebra), self-regulation feedback (e.g. "Check your steps") is most effective for problem-solving. Feedback that focuses solely on grades or praise (d = 0.04) hinders motivation in maths.

Kingston, N., & Nash, B. (2011), in a meta-analysis, found formative feedback in maths (e.g. quizzes with comments) had a moderate effect size (d = 0.48).

Immediate feedback (e.g. during practice problems) is twice as effective as delayed feedback for skill mastery, and effects were stronger for low-achieving students (d = 0.58) than high achievers (d = 0.26).

Wisniewski, B., Zierer, K., & Hattie, J. (2020), in a meta-analysis, found verbal feedback from teachers (d = 0.64) outperformed written feedback (d = 0.37) in maths.

Feedback was most effective when it reduced cognitive load (e.g., focused on 1–2 errors at a time), included worked examples (e.g., showed correct solutions to word problems), and focused on task mastery (e.g. "Try breaking this equation into steps").

Kluger, A. N., & DeNisi, A. (1996), in a meta-analysis, found 33% of feedback interventions in maths decreased performance, typically when feedback was overly complex or compared students to peers (e.g. "You scored below the class average").

Fyfe, E. R., & Rittle-Johnson, B. (2016), in a meta-analysis, found feedback's impact depends on prior maths knowledge. Novices benefit from direct corrective feedback (e.g. "This step is wrong because..."), and experts benefit from guided feedback (e.g. "What's another way to solve this?"). "Try again" feedback (without explanations) hinders learning (d = -0.21).

Hattie's effect size for giving and receiving feedback was 0.51 with a robustness of 4.

Characteristics of impactful feedback

Feedback is most impactful when it's timely, personal, actionable, linked to learning goals, does not compare to the performance of other students, and is tailored to students' existing understanding.

Timely feedback helps students identify and correct misunderstandings before they compound and guides future study strategies by pointing out issues and suggesting targeted practice.

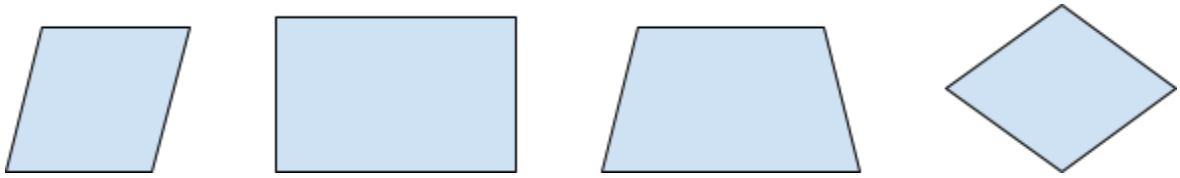
Personal feedback shows that teachers care about individual progress through comments on their responses and helps them to understand how to improve. It also makes students feel supported so that they seek help when needed. It may also encourage self-reflection (e.g. "Why did I make this error?") and help them to evaluate their problem-solving methods.

Examples of Questions on Different Skills and Abilities

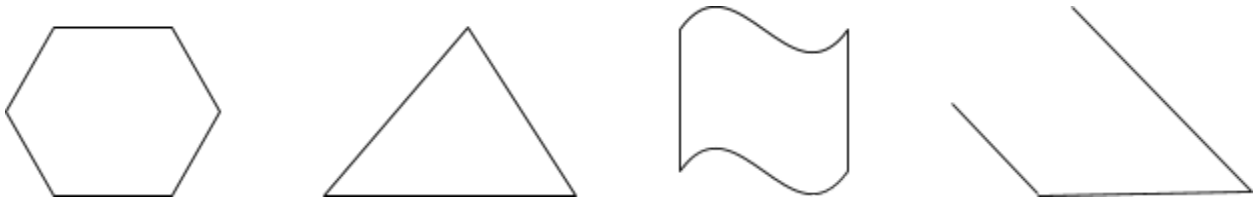
Understanding of concepts

Recognition of examples and non-examples

1. Which of the following are parallelograms



2. Which of the following figures are not polygons? State the property that prevents it from being a polygon.



3. For the following numbers, write W if it is a whole number, F if it is a fraction, I if it is an integer, R if it is a rational number, or IR if it is an irrational number. If more than one of these is true, write all of them.

(a) 6 (b) $\frac{3}{7}$ (c) -5 (d) $-\frac{5}{8}$ (e) $\sqrt{2}$

Multiple representations of concepts:

Model the following in as many ways as you can, e.g.

1. 4×5.
2. $\frac{3}{4}$.
3. A linear equation $y = 2x + 3$,

Deep processing of concepts

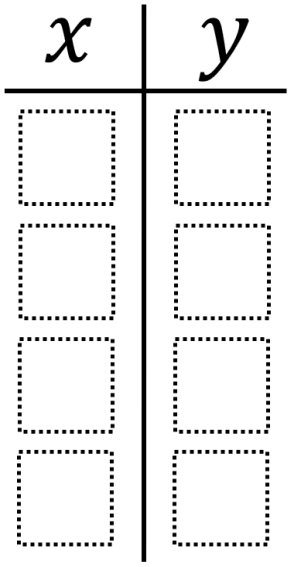
Open-Ended Questions

Ask questions that have many correct answers and follow it by writing all the answers and a discussion, e.g., "Are these answers correct?", "How did you find the answer?", "Which strategy is most efficient?"

1. Using digits 1-9, each digit once, fill in the boxes so that the statement is true.

   $\square + \square = \square - \square$

2. Compute mentally 56 - 19, and describe the strategy you used.
3. A decimal number with tenths was rounded to 6; what the number may have been?
4. Write 21 as a result of some mathematical operations on whole numbers.
5. Write a fraction less than $\frac{1}{4}$. How do you know it is less than $\frac{1}{4}$?
6. Give examples of how percentages are used in daily life.
7. Draw a rectangle whose area is 12 cm².
8. Using the digits 1 to 9 at most one time each, fill in the boxes to create a table of values that represent a linear function



Source: www.openmiddle.com

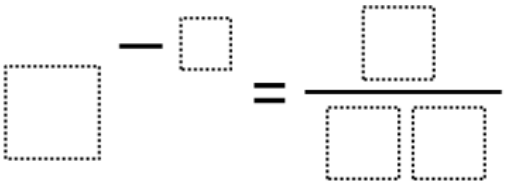
9. Write a quadratic equation, that has

   (a) Two real roots

   (b) Two equal roots.

   (c) No real roots.

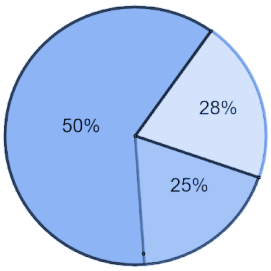
10. Use the digits 0 to 9 at most one time each; place a digit in each box to make a true statement.



(Answers are $3^{-2} = \frac{1}{09}$, $7^{-2} = \frac{1}{49}$, $8^{-2} = \frac{1}{64}$, $and\ 2^{-3} = \frac{1}{08}$.)

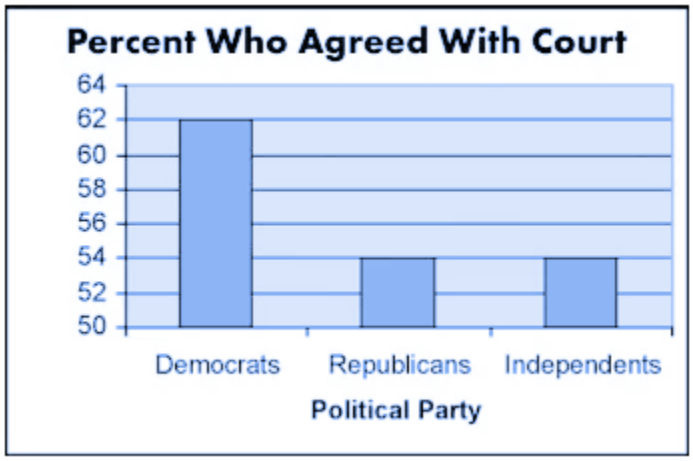
Source: www.openmiddle.com

11. What is wrong with the pie chart given below?



Answer: (i) The percentages do not add to 100.
(ii) 50% is represented by more than half the circle.

12. CNN, an American newspaper, used a graph similar to the one below to show who agreed with the decision to remove the feeding tube. Why is it misleading?



Answer: The y-scale starts at 50, which gives the impression that the number of Democrats who agree to remove the feeding tube are three times that of Republicans or Independents, whereas the difference is only 8%.

13. Some students gave these answers to different questions. How do you know these are incorrect?

(a)
$$\begin{array}{r} 49 \\ +33 \\ \hline 712 \end{array}$$

(b) $\frac{2}{3} + \frac{3}{5} = \frac{5}{8}$

(c) $3x - (x - 5) = 2x - 5$

(d) $(a + b)^2 = a^2 + b^2$

Relation between concepts: Ask questions about relationships between different concepts, e.g.

1. If 23 + 48 = 71, find 48 + 23, 71 - 48 and 71 - 23 in your head.

2. Fill in the blank cells so that every row, column and diagonal adds to 15.

8	1	
3		7
	9	

3. Fill in the missing numbers in the following multiplication table from 2 to 9, in which rows and columns are shuffled.

X	5				6			
2								
		40						
				49				
	20					36		
		72						
			9					12
					48			

Source: mashupmath

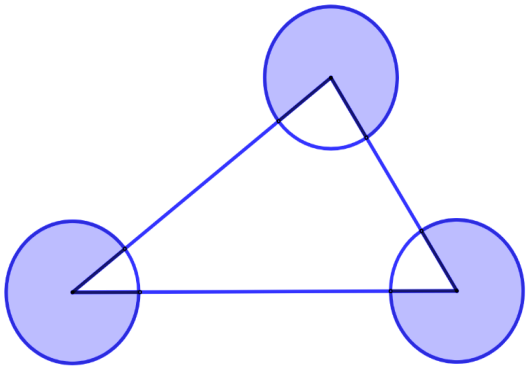
4. How are a square and a rhombus similar and different?

5. Is a rectangle always a square? If not, draw a rectangle for which it is not true.

6. How can you verify that the factors of $(2x^2 - x - 3)$ are $2x - 3$ and $x + 1$?

7. Design an experiment to show the relationship between the length of a pendulum and its period.

8. The triangle given below has a circle with a radius of 2 cm centred at each vertex. Find the area of the shaded region.



(Answer: 10π.)

Applications of maths concepts to real-world scenarios

1. Suggest where you find numbers and shapes in your surroundings.

2. Suggest situations where percentages, ratios and proportions are used in real life.

3. Give examples of variables in real life that vary directly and inversely.

4. Solve word problems to apply operations or a combination of operations on numbers to real-world situations.

5. Solve word problems involving variables that vary directly and inversely in real life and other subjects.

6. Write stories for which you will use an operation or combination of two or more operations.
7. Give examples of real-world situations where we use **'ish'** numbers.
8. Give examples of situations in real life where we round off numbers.
9. Give examples of real-life situations where we use an enlargement or reduction of objects.
10. Write and solve algebraic equations relating to problems in real-world situations.
11. Draw graphs relating to two variables, e.g., cost of taxi fare and distance travelled.
12. Estimate the paint required to paint a room with given dimensions.
13. How will you determine the height of a tree?

Applications of maths concepts to other subjects

1. Make a person, animal, or some object with tangram pieces.
2. Tessellate by making tiles of paper using Escher's method.
3. Find the distance between towns on a map in geography.
4. Apply direct and inverse proportions to problems in physics.
5. Apply functions to economics, physics and other subjects

Knowledge of facts, vocabulary, and fluency of procedures

Select examples from the textbook. Most of the books have questions on these arranged in order of difficulty. Add questions on fluency, e.g. mental math requiring composition and decomposition of numbers, associative laws, or use of formulae to simplify computations

Communication Skills

Discussion

Follow all open and rich questions by discussion

1. Create a pattern using chips of 2 colours, and explain your strategy.
2. Calculate mentally 36 + 47, and explain your strategy.
3. How does a shopkeeper, a farmer or a housewife use maths in his/her work?
4. Give examples in daily life where fractions, decimals or percentages are used.
5. How are a square and a rhombus similar and different?
6. A rectangle has a perimeter of 20 cm. What could its sides be (use whole numbers only)?
7. Make a tessellation in your notebook by altering a rectangle using the Escher method.
8. A parallelogram has an area of 24 cm^2 . What could its base and height be? (use whole numbers only)
9. If you cut a cube by a plane, what shapes would you get? Why do you say that?
10. Where are negative numbers used in daily life?
11. Give examples of algebraic identities. How do you know it is an identity?
12. Discussion for presentation of solutions to rich tasks will be given in rich tasks.

How teachers may evaluate a discussion

After a discussion, a teacher may note using a checklist, whether the students

- Listened actively.
- Disagreed respectfully.
- Participated
- Asked good questions.
- Gave good answers.

Examples of exit ticket entries:

- What did you learn today?
- What part of today's lesson was most challenging for you?
- What is one question you still have about today’s topic?
- How is it related to some earlier topic?
- How may it be applied in real life?
- I used to think …, but now I think….

Teachers may give it after a lesson in class.

Examples of journal entries

1. How does a bank use maths?
2. Give examples in real life where fractions, decimals or percentages are used.
3. Create a concept map that shows the relations between special quadrilaterals and write a description of that in words.
4. Give a situation in which a linear model is appropriate. Write an equation for it, draw its graph, and make a table.
5. Give examples of two simultaneous equations that have one, no, or an infinite number of solutions.

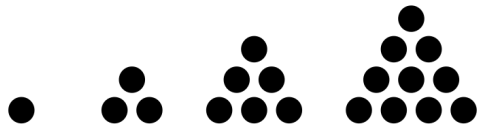
6. Write a report of the problem solved in a group that
    - Gives all steps for solving the problem.
    - How do you know your answer is reasonable or correct?
    - What did you learn from discussions in your group and the presentation and discussion of solutions by other groups? How you plan to use it in future.
    - How can this problem be further extended or explored?

Abilities and skills for higher-order thinking

Ask questions requiring recognition of patterns, intuition, divergent thinking, inductive, deductive, and spatial reasoning.

Recognition of patterns

Continue the series given below for the next two terms. State the rule for its formation. How many dots will the n^{th} term have?



.....

Use concepts to develop mathematical intuition

1. If you cut a pizza into more slices, will each slice be smaller or larger?
2. Multiplying by a number less than 1 makes the result smaller or bigger for
   (a) positive numbers?
   (b) negative numbers?
3. Will the graph open downwards or upwards? Why?
4. In any triangle, the longest side is opposite which angle?

5. How can you increase the reliability of an estimated proportion or mean for a population?

Spatial thinking

Ask students to

- Create shapes with tangrams.
- Create shapes congruent to a shape on a geoboard or dot paper.
- Identify and create shapes that will tessellate.
- Transform shapes by translation, reflection and rotation.
- Identify if a shape is the translation, reflection, or rotation of a given shape.
- Create shapes that are enlargements or reductions of a given shape
- Create and identify nets of 3D shapes.
- Identify the shape of intersections when a plane intersects a 3D shape in a horizontal, vertical, or slanting manner.
- Estimate distances between towns from maps.

Logical thinking

1. Using digits 1 to 9 at most one time each, place a digit in each box to give the greatest and least possible value of x as a solution.

   $\square\, x + \square = \square$

2. The sum of five consecutive numbers is a multiple of 5. Prove this statement or give a counterexample.

3. Find the number of 1×1 m² tiles you will need to put around a square swimming pool of size 8×10 m².

4. In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half of the lake?

Inductive reasoning

Give many examples of a law, and ask students, 'What do you notice?' Express it as a rule in words.

1. Commutative law in arithmetic or algebra.
2. Sums of 3 consecutive odd numbers.
3. Angles made on a circle by a chord.
4. Describe a rule given a few terms of a series.

Deductive reasoning

Give laws and ask students to apply it to particular examples, e.g.,

1. Any odd number can be expressed as 2n + 1; express 19 in this form.
2. Give a theorem in geometry and ask them to apply it to a particular problem.
3. Prove theorems in geometry.
4. The number system we use has base 10 (uses 10 digits from 0-9), so we can express any number as a sum of multiples of powers of 10, e.g.,

   $347 = 3 \times 10^2 + 4 \times 10 + 7.$

   The computers use a number system with base 2 (i.e., use 2 digits, 0 & 1 only). In that system, we can express any number as a sum of multiples of powers of 2, e.g., 1 = 1, 2 = 10 = $1 \times 2^1 + 0$, 3 as $11 = 1 \times 2^1 + 1$, 4 as $100 = 1 \times 2^2 + 0 \times 2 + 0$.

How would you write the following numbers in base 2?

(a) 5 (b) 6 (c) 7

Ans. (a) 5 = 101 (b) 6 = 110 (7) 111

Examples of problems using different problem-solving strategies

These problems can be solved using other strategies too.

1. **Use hands-on objects or role-play the situation**

   Ask students to divide 10 objects into two groups and write a number sentence for it. What strategy would you use to check that you have not missed any combination?

   Give 12 counters or sticks and ask them to divide it equally among 4 students and write a number sentence for it.

2. **Guess and Check (with refinement)**

   This isn't just random guessing. It involves making an educated guess, checking if it works, and then using the result to refine your next guess.

   Example: I am a two-digit number; I have a 5 in the unit's place; I am greater than 40 but less than 50. What number am I?

3. **Make a Table:** Organising information systematically can help reveal relationships or possibilities.

   Example: How many different ways can you make 25 rupees using one-rupee, five-rupee and ten-rupee notes?

4. **Create an organised list.**

   Example: Five players – A, B, C, D & E – enter a badminton tournament in which each player must play every other person exactly once. Determine the total number of games that will be played.

5. **Simplify the problem**

If a problem seems too complex, try solving a simpler version of it first. This can help you identify the core principles or develop a strategy that can then be applied to the original problem.

Example: How many diagonals does a polygon with 20 sides have?

6. **Problem solving with algebra**

Taxi A charges a flat fee of 25 INR per km, and Taxi B charges INR 100 + INR 20 per km.

(a) Is there a mileage at which the charges would be the same for the two taxis?

(b) For what distances does taxi service B cost less than taxi service A?

7. **Make a Drawing or Diagram**

A farmer wants to make a barbed wire hedge around a square that has 10 poles at equal distances on each side. How many poles does he need?

8. Work Backwards

This strategy is effective when the end result is known and you need to find the starting point or a previous state.

Problem: Alice bought a book for INR 50 and then spent half of her remaining money on a dress. She had INR 100 left. How much money did she start with?

9. **Use Logical Reasoning**

Strategy: Eliminate impossible answers and use logic.

Example: A number is divisible by 3 and 5 but not by 10. What could it be?

10. **Look for a Pattern**

    Example: Find the sum of the first few odd numbers. What pattern do you notice?

For solutions of these problems using these strategies, see Appendix 3.

Rich questions to promote higher-level thinking

Rich questions that can be solved by using many problem-solving strategies and have low floors and high ceilings should be selected and given in small groups to provide opportunities for peer-to-peer communication, learning diverse strategies, and developing skills for teamwork.

1. I am thinking of a number. When I divide it by 3, 4 and 5, I always get 1 as a remainder. What is my number? What other numbers would give me 1 as a remainder when I divide them by 3, 4 and 5?

2. Find the totals of a few sets of 3 consecutive odd numbers.

   (a) Is there anything special about all the totals?

   (b) Is there a quick way to work out the total?

   (c) Can you predict what would happen if the numbers were even numbers?

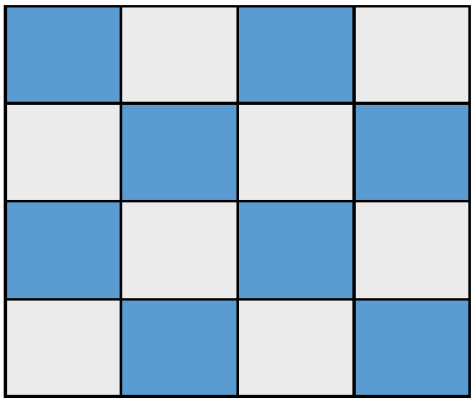
3. Many numbers can be expressed as a sum of two or more consecutive numbers, e.g. 5 = 2 + 3, 6 = 1 + 2 + 3, 10 = 1 + 2 + 3 + 4, 11= 5 + 6. Are there some numbers that cannot be expressed as the sum of two or more consecutive numbers? If yes, can you make a conjecture about them? (All numbers except for powers of 2 are sums of consecutive numbers).

4. I have a special number; if I add the sum of its digits to the product of its digits, I get my original number.

(a) What could my number be?
(b) Can you find more than 1?
(c) Can you find them all?
(All two-digit numbers ending in 9 using a little algebra)

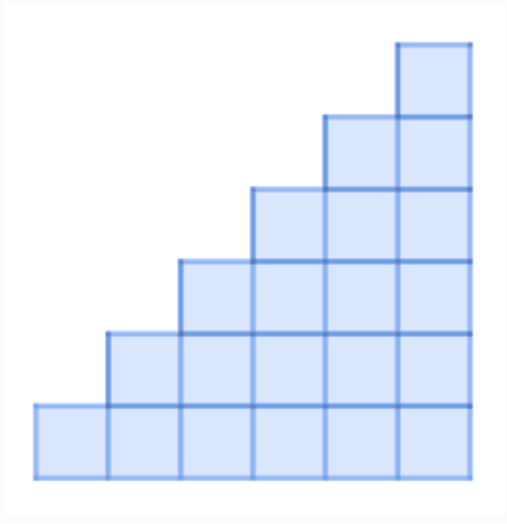
Source: enrichmaths.com

5. How many squares are there on a 4×4 chequerboard?



Answer: 30.

6. A triangular array of squares can represent triangular numbers.



(a) Imagine two copies of a triangular array of squares. Can you picture how to fit them together to make a rectangle? If yes, what is special about the dimensions of the rectangle?

(b) What do you notice about doubling triangular numbers?

(c) Experiment with different triangle numbers and explain what is special about the rectangles made from two identical triangle numbers.

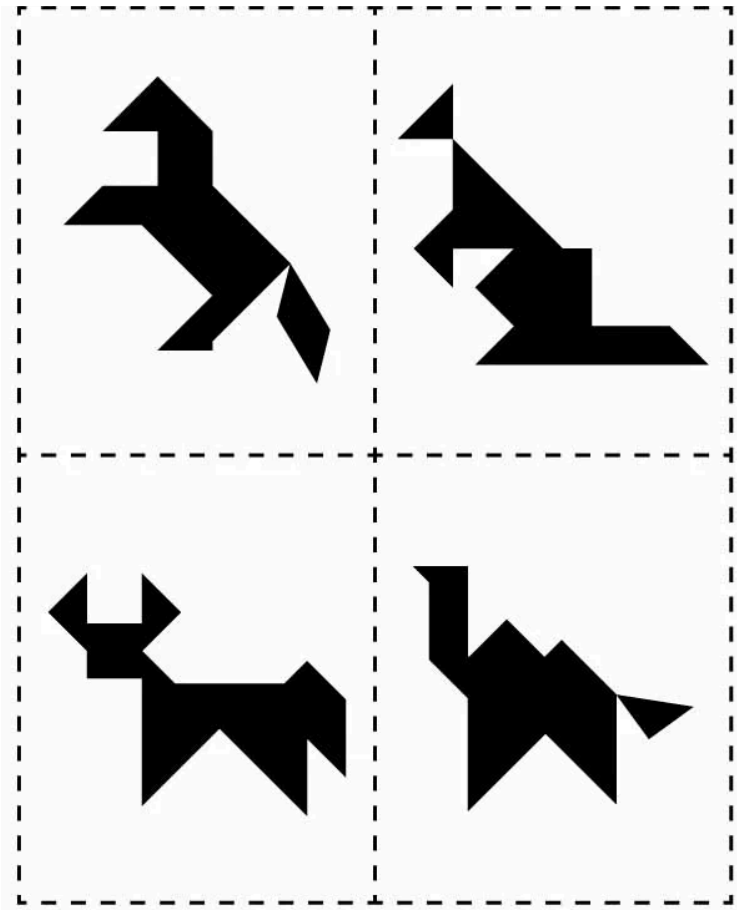
(d) Can you write down the dimensions of the rectangle made from two copies of the 250^{th} triangle number?

(e) Deduce a strategy for working out the n^{th} triangle number.

(f) How can you decide if a number, e.g., 4851, is triangular or not?

Source: enrichmaths.org

7. Make objects using all tangram pieces e.g.



8. Rob and Jennie were making necklaces to sell at the school fair. Each necklace was to have eight beads, four of one colour and four of another. They decided to make them symmetrical, like this.

(a) How many different symmetrical necklaces could they make with these beads?

(b) Can you find them all?

(c) How do you know you haven't missed any out?

(d) What if they had 9 beads, five of one colour and four of another?

(e) What if they had 10 beads, five of each?

Source: nrich.maths.org

9. The perimeter of a triangle is 20 cm. Using whole numbers, how many sets of side lengths using numbers 1 to 9 can you have?

10. (a) Use tangram pieces to make a square with 3 pieces.

(b) If the square piece represents 1 unit of area, find
(i) the areas of squares made in (a).
(ii) perimeters of squares made in (a).

(c) If the square piece represents 1 unit of area, what is the areas of the figures made in Q.7?

11. If you continue folding a paper in half,

(a) How many layers are being made with each fold?

(b) Write a function that relates number of folds (x) to number of layers (y)

(c) Draw a graph of that function.

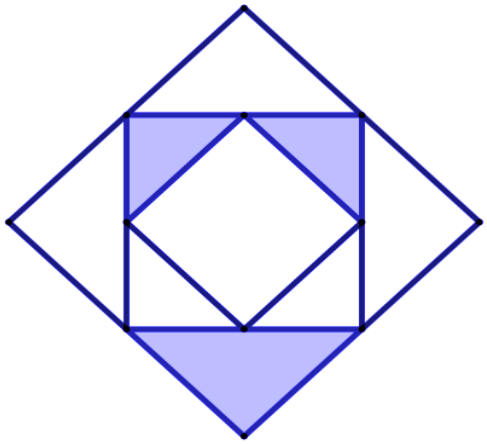
(d) How many layers will there be if you folded 3 papers?

12. A teacher took a survey of students to find the kind of pets they have at home and found each student has at least one pet. He found that 20 students have a cat, 15 students have a dog, and 7 students have a bird. If 2 students have a bird and cat, 4 students have a bird and a dog, 6 students have a cat and a dog, and 1 student has all 3 pets. How many students are in his class?

(The answer is 31 students. Hint: Use Venn diagrams and work backwards.)

Source: Titu AndreesScu et al. (2020)

13. The diagram given below shows some squares whose sides intersect other squares at the midpoint of their sides. The shaded region has a total area of 4 cm². Find the area of the largest square.

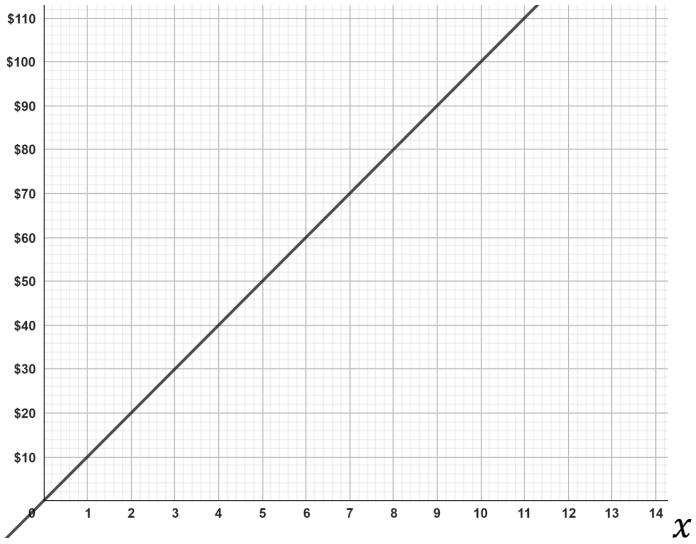


(Answer is 32 cm^2.)

If another larger square was drawn so that the vertices of the largest triangle were the midpoints of the sides, what would be the area of that?

Source: Purple Comet! maths Meet 2019 contest. http://purplecomet.org

14. Anita is saving every week INR 10 to purchase a blouse costing INR 100. The graph shows the amount she will have saved at the end of different weeks.



Week

(a) Write the equation of the line represented in the graph

(b) Describe how the components of the equation represent the situation.

(c) (i) What would happen to the graph if she already had INR 20 saved?
(ii) Represent this situation on the graph provided.
(iii) Write the equation of the line represented in the graph.
(iv) Describe how the components of the equation represent the situation.

(d) (i) What would happen to the graph if she saved INR 20 a week?
(ii) Represent this situation on the graph provided.
(iii) Write the equation of the line represented in the graph.
(iv) Describe how the components of the equation represent the situation.

(e) (i) How would the graph change if she already had INR 20 saved and started saving INR 20 a week?
(ii) Represent this situation on the graph provided.
(iii) Write the equation of the line represented in the graph.

(iv) Describe how the components of the equation represent the situation.

15. Drop a Rubik's Cube into the paint so that it is completely covered. When the paint is dry, imagine smashing it on the floor and breaking it apart into smaller cubes.

(a) How many of the cubes have one face covered in paint?

(b) How many cubes have two faces covered in paint?

(c) How many have three faces covered in paint?

(d) How many have no faces covered in paint?

(e) What about a 4 x 4 x 4 cube?

(f) n x n x n cube?

17. Form groups of 3-4 students and ask them to collect data for a question of interest to them (e.g., the kind of programmes they like to watch on TV, scores on a test) from students in their class, organise it, present it visually, decide about summary measures to compute, interpret and present their findings to the class.

18. Use puzzles and games

Puzzles help students experience the beauty and joy of mathematics as a subject of challenge, wonder, and discovery. They provide meaningful opportunities for students to think deeply, work collaboratively, and develop a lifelong love for problem-solving.

Games make maths engaging and involve problem-solving and strategic thinking, and they foster deeper understanding and positive attitudes.

For sources for puzzles and games without the use of technology, see Gaskin, D. (2015-2021). Math you can play series: Books 1-4.

Loveformaths.org

https://www.weareteachers.com/maths-card-games/

https://www.mathsgiraffe.com/blog/23-ways-to-use-dice-in-maths-class

Evaluation of Rich Tasks

- Use rubrics to outline the expectations for students.
- Circulate throughout the room during activities and observe and listen to group interactions to gain insights into students' understanding and problem-solving skills and social skills for collaboration.
- Ask them to make a presentation to the class. It will help them develop communication skills and learn different strategies for problem-solving.
- During presentations, ask or encourage students to ask questions such as,
  "Describe the approach you used to solve the problem?"
  "How does your approach differ from that of another group?"
  "Which approach, if any, do you now think (after all groups had presented their approaches) would have been better to use?"
  "How do you know your answer is reasonable?"
  "How will you convince yourself, peers or a sceptic that your answer is correct?"
- Ask each of them to write it in a journal and collect student work (e.g., recordings of their thinking, solutions) and evaluate their understanding for future instruction.
- Identify areas of strength and weakness, and set learning goals for improvement

Grading

If the school requires grades at the end of a year or a semester, a summative test, a student's self-assessment, and some components of formative evaluation, such as journal entries, may be used.

Key Takeaways

Tools for conducting formative assessment

Teachers can conduct formative assessment by

1. **Written assignments:** These can be given in class or as homework, low-stakes quizzes or exit tickets and are especially valuable for providing opportunities for
    - Practice and spaced practice of procedures.
    - Writing mathematical proofs.
    - Reflecting on learning experiences in an exit ticket or journal.
    - Synthesising learning, e.g. how a shopkeeper, a farmer, or a housewife uses maths in daily life.
    - Solution of open-ended questions e.g. write two simultaneous equations that have no, one, and an infinite number of solutions.
    - Solutions to rich questions solved in groups in class and reflections on the experience.
2. **Discussion:** It is especially valuable for discussion of open-ended and rich questions solved in groups in the whole class.
3. **Self-assessment:** It empowers students to reflect on their learning, identify strengths and areas for improvement, and take ownership of their progress. Fostering self-assessment skills helps students develop metacognition, critical thinking, and independence.

    Exit tickets and journaling with a rubric can be used for self-assessment.
4. **Peer assessment:** It can be done by providing rubrics for assessment to students.

It is helpful for learning from peers and developing the ability to analyse and think critically.

It saves time for the teacher.

5. **Observations:** These provide valuable insights to the teacher into students' understanding and tailoring instruction to meet individual needs and helping students achieve their mathematical potential.

   These are necessary when using manipulatives and for evaluating students' interactions in collaborative settings.

Providing Impactful Feedback

Feedback is most impactful when it's timely, personal, actionable, linked to learning goals, and does not compare to the performance of other students.

Challenges in Conducting Formative Evaluation

The greatest challenge is time constraints: To overcome time constraints

- Ask students to keep 1, 2 or 3 fingers against the chest to indicate they need more coaching, sometimes make mistakes, or have learnt it well for feedback on a lesson.
- Use tools like Google Forms for instant scoring of quizzes and data analysis if technology is available.
- Use peer assessment for journal writing by giving students rubrics.

7

Professional Development of Mathematics Teachers

Teachers are the cornerstone of effective education. They play a central role in shaping individuals, societies, and the future. They are the backbone of the education system, influencing not only academic success but also personal and social development.

Present Status of Professional Development in India

Professional development is crucial for empowering teachers and school leaders with modern teaching strategies, leadership skills, and student support techniques

Table 7.1 highlights the details of teachers and school leaders participating in the various professional development programmes in PARAKH's Rashtriye Survekhshan (NCERT, 2024).

Table 7.1: Present Status of Continuous Professional Development (NCERT, 2024)

Programme	Percent Participated
Schools organised teacher training sessions/workshops for parents around 'understanding their children better/managing their behaviour'.	86 %
Schools organised any teacher training session/workshop on behavioural management in a classroom in the past two academic years.	81 %
Schools organised teacher training sessions/workshops on managing and understanding adolescent behaviour in the past two academic years.	78 %

Schools organised teacher training sessions/workshops on mental health awareness in the past two academic years.	75 %
Schools offering any regular professional development sessions for teaching staff in the past two academic years.	67 %
Schools organised teacher training sessions/workshops on managing emotions/emotional intelligence in the past two academic years.	64 %
Schools organised teacher training sessions/workshops on stress management in the past two academic years.	61%
School principals/head teachers undergone training on **HPC**[1].	51%
School principals/head teachers have undergone school leadership training programmes organised by NIEPA or state governments.	43%
Teachers participating in the 'mentoring and/or peer observation and coaching as part of a formal school management in BRCs/CRCs' activity in the last 12 months.	39%
Teachers participated in the ICT-related training during the last 12 months.	38%
Teachers Participating in Professional Development Programmes in the Last 12 Months.	31%

It is good that schools are organising programmes for teachers. But most of these training programmes for teachers and heads of schools have related to the management of emotions and behaviour. And the data was collected on only on percentage of teachers participating in it and not its effectiveness.

There is a need to conduct training programmes on effective teaching strategies, formative evaluation and feedback.

[1] The HPC (Holistic Progress Cards) is an integrative and comprehensive document of the learners' progress captured across several curricular elements based on their performance on the competency-based and multidisciplinary activities.

Need for Professional Development

As pointed out in NEP (2020) and corroborated by poor results in CTET and state TETs, there is a need to improve maths education and the training programmes for teachers. The curriculum for teachers' preparation should include deeper understanding of maths (e.g. Nolan, E. C., 2016; Dixon, J. K., 2016); teaching strategies for deeper understanding of concepts and higher-order thinking, formative evaluation and impactful feedback based on empirical evidence and the importance of **action research** and how to conduct it.

Continuous professional development for teachers by in-service training is also necessary for improving the quality of teachers and adapting to changing educational policies and job markets and incorporating the latest research findings.

Characteristics of Effective Professional Development (PD) of Teachers

PD refers to educators actively improving their knowledge, skills, and competencies through learning opportunities to become more effective in the classroom by staying updated on new content and recent research on pedagogy.

Darling-Hammond, L., Hyler, M. E. and Gardner, M. with assistance from Espinoza, D. (2017), based on a review of studies on PD, identified seven widely shared features of effective professional development. It

1. Is focused on a subject.
2. Incorporates active learning by using authentic artefacts, interactive activities, and other strategies to provide deeply embedded, highly contextualised professional learning.
3. Supports collaboration for teachers to share ideas and create communities that can positively change the culture and instruction of their entire grade

level, department, school or district.

4. Uses models of effective practice to provide teachers with a clear vision of what best practices look like. Teachers may view models that include lesson plans, unit plans, sample student work, observations of peer teachers, and video or written cases of teaching.

5. Offers opportunities for reflection.

6. Is of a sustained duration.

Yoon, K.S. et al. (2007), in a meta-analysis of 1,300 studies (9 rigorous experimental studies included), found that PD programs with at least 14 hours of training showed significant improvements in students' maths scores (d ~ 0.54). Sustained PD (over weeks/months) had a stronger impact than one-time workshops.

Timperley, H. et al. (2007), in a meta-analysis of 72 studies (maths-focused PD), found an effect size (d ~ 0.62) on student maths scores. The most effective PD programmes were the ones focused on specific maths content (e.g. fractions, algebra), active learning (lesson study, peer coaching), and provision of follow-up support (coaching, feedback).

Gersten, R. et al. (2014), in a meta-analysis of 32 high-quality maths PD studies (K-12), found an effect size (d ~ 0.47). The content-focused PD (deepening teachers' maths knowledge) had the strongest impact, and coaching and modelling were more effective than lectures.

Lynch, K. et al. (2019), in a meta-analysis of 95 studies (maths PD interventions), found an effect size (d ~ 0.42). Longer PD duration (30 or more hours) led to better outcomes, and job-embedded PD (e.g. in-class coaching) was more effective than off-site training.

Kennedy, M.M. (2016), in a meta-analysis of 28 maths PD studies, found an effect size (d ~ 0.50). Programmes that combined content knowledge and pedagogy (e.g. how to teach fractions effectively) had the highest impact, and

collaborative PD (e.g. teacher study groups and professional learning communities (PLCs)) improved maths instruction.

Hattie, J. (2023)'s effect size was 0.59 with a robustness of 3 for both the teacher attributes and teacher education.

Benefits of Professional Development of Teachers

1. PD can introduce teachers to teaching methods and technologies that can make learning more engaging and effective for students.
2. PD can help teachers deepen their understanding of the subject matter and stay current with the latest research and developments in their fields.
3. PD can equip teachers with the skills to differentiate instruction to meet the diverse needs of all learners in their classrooms.
4. PD can help teachers refine their classroom management, communication skills, and assessment practices.
5. PD can increase teachers' confidence in their ability to address challenges in the classroom.
6. PD can help teachers feel more supported and valued, reducing burnout and increasing job satisfaction.
7. PD can ensure teachers have the knowledge and skills to adapt their instruction to evolving curriculums and educational standards.
8. PD can help teachers analyse student data to identify areas for improvement and make data-informed decisions about their instruction.

Resources for Professional Development of Teachers

1. **Workshops and seminars:** These are focused training sessions on specific topics such as how to differentiate instruction, manage the classroom, or use new technological tools. Many such workshops are conducted by organisations such as NCTE, NCERT, Regional Institutes of Education

(RIE) and SCERTs, and the National Initiative on Mathematics Education (NIME). Many such workshops are also available on YouTube.

2. **Conferences and presentations:** These provide information about discussion and research on specific topics. These also provide opportunities to network with other educators.

   Teachers can attend conferences conducted by NCERT, the Indian Mathematical Society, the Mathematics Teachers' Association (MTAI), the International Science Festival (IISF) – Maths Track, the National Conference on Mathematics and Computer Science (NCMCS), the Ramanujan maths Society Conferences, the Indian Institutes of Science Education and Research (IISER), and the International Conference on Technology in Mathematics Education (ICTMA) – India Chapter, and some universities and IITs.

3. **Mentoring programmes:** Pairing experienced teachers with newer teachers to provide guidance and support.

4. **Online courses and modules:** Accessing learning materials through online platforms such as

   - Diksha (diksha.gov.in): It offers free courses on competency-based maths pedagogy, formative assessment, and STEM integration

   - Nishtha (https://itpd.ncert.gov.in/mod/page/view.php?id=504**):** Nishtha provides the latest pedagogies regarding foundational literacy and numeracy, formative and adaptive assessment of learning outcomes, competency-based learning, and related pedagogies, such as experiential learning, arts-integrated, sports-integrated, and storytelling-based approaches, etc., for all stages of education.

   - Swayam (https://swayam.gov.in): Swayam hosts courses which provide video lectures, reading material that can be downloaded/printed, self-assessment through tests and quizzes and an online discussion forum for clearing doubts. Steps have been taken to enrich the learning

experience by using audio-video and multimedia and state-of-the-art pedagogy/technology.

- Coursera (Coursera.org): It hosts courses in maths and maths education provided by professors from top universities from many countries. Teachers can try to get financial aid or audit the courses for free (when you audit a course, you have access to all videos and tests. However, most of these will not provide feedback on tests).

- edX (www.edx.org): It hosts courses in maths and maths education provided by professors from top universities. Teachers can get financial aid or audit the courses for free.

- **Videos:** Teachers can watch videos provided by

    - e-Pathshala and Central Institute of Educational Technology (CIET). It provides free digital textbooks, video lectures, and interactive maths modules in multiple languages.

    - Khan Academy has a vast library of video lessons, practice exercises, and articles covering maths topics from basic arithmetic to calculus. It is also available in Hindi and several other Indian languages.

    - State SCERT Portals

    - nctm.org: Most of their resources are available to members, for which they charge a fee. However, they have some good publications on the internet also.

    - Illustrative mathematics provides high-quality maths tasks, lessons, and assessments aligned with the Common Core State Standards (USA).

    - https://www.nysed.gov/edtech/digital-content-resources-mathematics: New York State Department of Education provides high-quality

maths tasks, lessons, and assessments aligned with the Common Core State Standards.

- YouTube videos: For a selection of YouTube videos for classes V-VIII based on the NCERT syllabus and High school Algebra, Geometry, and Statistics with exercises based on Common Core State Standards (US); see my presentations

- **Class V**
  https://docs.google.com/presentation/d/14umceabPLhLrjD9ZN3Fe2eUufgVO30HnQj4Z8q1EvKk/edit?usp=sharing

- **Class VI**
  https://docs.google.com/presentation/d/1jJL3-maFDYDofF1qE2XeQmzcz2Lld1rzUOdDmAPCpEw/edit?usp=sharing

- **Class VII**
  https://docs.google.com/presentation/d/1HeqTxjlBAfT7mWXb2Vr_bLqT74vJfA0Qz2IGsoIRloU/edit?usp=sharing

- **Class VIII**
  https://docs.google.com/presentation/d/16hQ9j6Uzn0Yefh9OAHomdcpxjnrAUJ-fu1_tmbmQKJU/edit?usp=sharing

- **Algebra 1 for high School**
  https://www.researchgate.net/publication/378103133_Supplementary_Activities_for_Algebra_1

- **Geometry for high School using GeoGebra**

  https://docs.google.com/presentation/d/1lj4-7Ci5r9B5NG9jfbnMNSqdn1rruk6VXJrvXgSzKeM/edit?usp=sharing

- **Functions for high school using GeoGebra**
  https://docs.google.com/presentation/d/1212Z_GZnoTclf-m7p_iwdZaBJpIhzYYJ73WbPTvjA8E/edit?usp=sharing

- **Statistics for high school using GeoGebra**
  https://docs.google.com/presentation/d/13enOAEu8O5nu-JYlPeI5PkQsQEZtMfsFDwFOJa-IVvo/edit?usp=sharing

- **Use of Artificial Intelligence**

  Artificial intelligence can be used to generate lesson plans, worksheets, quizzes and tests for any topic and grade.

  How to use Google Gemini for teachers, see https://ditchthattextbook.com/google-gemini/?_bhlid=11d48047e35a14e50ebb54929bbca0806073634b

  For 50 AI tools for teachers, educators and classrooms (free and paid), see https://ditchthattextbook.com/ai-tools/

- **Books**
  Excellent books are available on teaching and evaluation of maths. The school libraries should be equipped with such books. For some suggestions, see further readings in resources.

- **Interactive Activities Using Technology**
  If students have access to computers, the following websites may be used by teachers for interactive activities.

  - **National Library of Virtual Manipulatives (NLVM):** Offers interactive simulations and activities for K-12 maths concepts, allowing students to explore and manipulate mathematical objects virtually.

- **Desmos:** An online graphing calculator with powerful features for exploring functions, creating interactive activities, and even building entire lessons.
- **GeoGebra:** A dynamic mathematics software platform for all levels of education that combines geometry, algebra, spreadsheets, graphing, calculus, and statistics.

  For how to integrate geometry with GeoGebra, see my monograph available free on Google Play:

  https://play.google.com/store/books/details/Harinder_Mahajan_Empowering_Geometry_Education_by?id=f4oREQAAQBAJ&utm_source=na_Med

Key Takeaways

- There is a need to improve teacher training programmes by incorporating teaching and assessment strategies that research has shown to be effective and deepening their understanding of maths, especially for primary teachers.
- Train teachers on how to organise data to look at students' performance and reflect on it for improving teaching.
- Schools should facilitate teachers' participation in seminars and workshops organised by government and non-government institutes.
- Schools should provide mentoring for beginning teachers.
- Educational leaders should support collaboration for teachers to share ideas and create communities that can positively change the culture and instruction of their entire grade level, department, school and/or district.
- Teachers should utilise resources such as Diksha, Nishtha, Coursera, edX and YouTube videos and webinars for improving their knowledge, skills, and competencies and staying updated on new content and recent research on pedagogy.

8

Key Findings and a Roadmap for Action

This final chapter synthesises the major insights from earlier chapters and presents a practical roadmap for implementing effective mathematics pedagogy in Indian schools, navigating the challenges and recommendations for stakeholders.

Synthesis of Major Insights

The present landscape of education in India suggests the need

- To build a strong foundation in numeracy.
- Move from rote learning to understanding and higher-order thinking.
- Use formative assessment to improve teaching and enhance learning.
- Provide feedback to students that is timely, individual, and actionable.
- Improving the quality of training programmes by including effective teaching strategies based on research and providing for deeper understanding of maths content for primary teachers in elementary teacher training programmes.
- Meet the shortage of teachers by attracting more able students to teaching by providing better emoluments and improving the quality of training programmes to enable more students to qualify in TET examinations.
- Provide continuous professional development of teachers to improve the quality of teachers.

There is not one way to teach mathematics or one instructional strategy that works for all learning objectives, for all students, and in all contexts, but there is compelling evidence for certain strategies or approaches to enhance attainment of different learning objectives.

Key Takeaways From Research in Education, Psychology and Neuroscience

Hattie's study (the research in education) suggests that a teacher's characteristics, e.g., his/her estimate of a student's achievement, credibility, clarity, not labelling students and teacher expectations, are highly effective for a student's learning.

Students' achievement also depends on how well the teacher aligns the curriculum content, learning intentions, success criteria, assessment and feedback in his teaching.

Among the teaching strategies, explicit teaching, direct instruction, the CRA approach and interventions to help students with learning difficulties were more effective in enhancing students' achievement.

Insights from research in maths education, psychology, and neuroscience suggest that the following approaches have high potential for enhancing learning.

- Engagement, retrieval, spaced practice, and interleaving for retention and consolidation of learning.
- Active approaches, e.g., the use of manipulatives, visuals, discussion, and applications to problems in daily life for engaging students and enhancing understanding of concepts and communication skills.
- Opportunities for collaborative or cooperative learning, especially for open-ended and rich questions followed by discussion for deep understanding of concepts, development of higher-order thinking, communication and social skills for teamwork, and learning from peers.
- Supportive classrooms, in which students are willing to take risks, see mistakes as opportunities to learn and good for the brain, and are not anxious.

- Formative assessment and feedback that is timely, individual, and actionable.
- Research from psychology also emphasised the importance of enhancing engagement, curiosity, motivation, self-efficacy, a growth mindset, and persistence for enhancing learning and tackling maths anxiety and procrastination, which hinder learning.
- Research in neuroscience on how the brain develops highlighted the importance of a progressive approach to teaching maths—starting with concrete, experiential learning in early childhood and gradually moving toward abstract reasoning in adolescence for optimising mathematical understanding and engagement at every age. This aligns with research in psychology too.
- The primacy and recency effects observed in memory highlight the need to teach new concepts at the beginning of a lesson and keep the duration of sub-learnings short (about 20 minutes duration) for retention.

The effectiveness of an approach also depends on the context. While choosing a new strategy, teachers should also rely on their own experience, knowledge of students, facilities available to them, and what resonates with them.

For any new approaches the teachers decide to implement, they should gauge the effectiveness of the approaches by using action research and share it with other teachers.

2. An Integrated Framework for Effective Mathematics Pedagogy

A successful pedagogy integrates learning objectives, instructional strategies, and assessment tools and provides for teacher support as given in Table 8.1.

Table 8.1: Integrated Framework for Effective Math Pedagogy

Learning Objective	Instructional Strategy	Assessment Tools*	Teacher Support
Deep conceptual Understanding	CRA approach; discussion, think, pair and share especially for open-ended questions	Observations; written assignments requiring multiple representation and connections of concepts; exit tickets; rubrics for discussion; and journaling	Provision of kits for manipulatives and facilities for providing handouts and opportunities for peer discussion or mentoring for teachers
Knowledge of Maths vocabulary	Explicit teaching, Use of retrieval, spaced practice and interleaving	Homework and quizzes	Provision of retrieval, spaced practice and interleaving in textbooks and PD
Procedural fluency	Guided teaching, practice, and spaced practice	Classwork or homework	Skill-specific PD modules
Communication in maths	Discussion	Discussion of open-ended questions; presentations or writing of solutions to open and rich questions in journals	Include questions in the textbook or handouts that are open-ended, rich, and require metacognitive thinking and emphasize these in PD

Problem-solving strategies	Explicit teaching for different problem-solving strategies and use of think, pair and share or discussion in class	Written solutions to problems using different problem-solving strategies and peer assessment	Include problems requiring different problem-solving strategies in textbooks or handouts
Rich questions	Collaborative learning after ensuring students have the prerequisites.	Presentations of group solutions of rich questions in class and their discussion; individual journaling of these, including questions on metacognitive thinking	Incorporation of these in PD and inclusion of these in textbooks or handouts

* For all assessments provide timely feedback and specific individual guidance that supports students in recognising misconceptions, refining problem-solving approaches, and building confidence in their abilities.

3. Addressing Challenges

Building a strong foundation in numeracy

The government has launched NIPUN and Padhe Bharat Barhe Bharat, programmes for building a strong foundation in numeracy. There is a need to monitor their implementation and evaluate them.

The approaches used by Agastya Learning Portal, Pratham, Jodo Gyan, and Edubridge have been successful in improving numeracy and may be emulated by other organisations.

Although this book has not covered preschool maths education, we would note that children's mathematical knowledge at school entry is the strongest predictor of later math success (e.g. Duncan, G.J. et al., 2007; Watts, T.W. et al, 2014 & 2018; Nguyen, T., et al., 2016; Mononen, R., et al., 2014). Similar findings have been obtained with respect to risk for math learning disability (MLD). Morgan, Farkas and Wu (2009) found that 70 percent of children who started and ended kindergarten below the 10^{th} percentile in mathematics were also below the 10^{th} percentile in 5^{th} grade, underlining the importance of early mathematical knowledge to later mathematical achievement. There is a need to ensure that pre-education programmes for ages 3-6 ensure mastery of knowledge and skills such as verbal counting, knowing the number symbols, recognising quantities, discerning number patterns, comparing numerical magnitudes, and manipulating quantities (i.e., adding and subtracting objects from a set).

(a) Limited resources

- Low-cost manipulatives developed by NCERT and non-government organisations can be used. If even that is not possible, many of them can be made by students, as suggested in Chapter 5.
- For visual representations, tables of 1-100, addition and multiplication can easily be incorporated in textbooks.
- As most students have a smartphone at home, teachers can refer students having difficulties or needing enrichment to view appropriate videos at home.
- Incorporating practices such as retrieval, spaced practice, interleaving, discussion and journaling in maths and collaborative learning in teaching does not need any additional resources and are very effective in enhancing learning. These should be incorporated

in teaching, pre-service and in-service professional development programmes.

- As most teachers do not have facilities for handouts and depend mostly on textbooks, questions requiring spaced practice and interleaving, requiring deep understanding of concepts, problem solving, higher order and metacognitive thinking Spatial reasoning, etc., should be incorporated in the books by NCERT and SCERTs, which are widely used. Similarly, tables of 1-100, addition and multiplication should also be provided in the textbooks.
- As most people have access to smartphones, involving parents and the community in students' learning would be helpful.

(b) Availability of teachers

- As teachers are the cornerstone of education, there is a need to meet the shortage of teachers, attracting more able students to teaching by improving service conditions, e.g., salary, provision of health insurance, and housing.
- There is a need to improve maths education and teacher training programmes so that more applicants can qualify in TETs.

(c) Quality of teachers

To improve quality of teachers, there is a need to

- Incorporate effective teaching strategies suggested by research in training programmes for teachers and professional development programmes for in-service teachers.

- As most teachers have at least a smartphone and a lot of resources are available on the internet and platforms created by government and non-government organisations in India, there is a need to make

teachers aware of them and encourage their use in pre-service and in-service programmes.

- Provide for deeper understanding of maths, especially for primary teachers in elementary teacher training programmes.
- Prioritise evidence-based teaching strategies in teacher training programmes and equip teachers to read, interpret, and evaluate research findings and carry out action research.

4. Recommendations for Stakeholders

Educational Policy Makers

- NEP (2020) and NCF (2023) articulate a progressive vision; there is a need to evaluate its translation by government institutes like NCERT and SCERTs in textbooks, assessment tools, teachers' guides, teacher training, and PD, and its implementation in classroom practice
- Meet the shortage of qualified teachers by providing better emoluments.
- Provide resources such as books on maths education and assessment, teaching aids, and maths labs for schools.
- Expand access to technology and its integration in maths education so that online resources for students and teachers can be used to improve teaching and learning of maths.

Educational leaders

- Facilitate participation in PD for in-service teachers
- Provide mentoring for new teachers by senior teachers.
- Provide opportunities for regular peer collaboration.

- Encourage teachers to network with teachers from other nearby schools; this is especially important for small schools.
- Reduce the teaching load and administrative work teachers are required to undertake so that they can try new practices, collaborate with teachers from their school or other schools to share and learn effective practices for teaching maths, and do action research.

Teacher Educators

- Integrate research-based strategies for teaching and assessment in the curriculum and use of feedback for improving instruction.
- Provide for deeper understanding of content in pre-service and in-service programmes.
- Make teachers aware of resources available online and encourage their use for further learning in teacher training programmes.
- Discuss the importance of a growth mindset in teachers and students.
- Provide guidance on how to carry out action research.

Teachers

- Implement research-based effective teaching practices.
- Use formative evaluation.
- Reflect on teaching and assessment by using evidence of students' learning.
- Provide timely, individual and actionable feedback.
- Collaborate with other teachers to learn from each other and share experiences.

- Utilise PD opportunities and resources available online for learning, teaching, providing for individual differences, and tools for formative evaluation.
- Evaluate implemented strategies and revise or share it depending on its effectiveness**.**
- Join online groups of maths teachers to share and learn from each other's experiences.
- Carry out action research whenever trying a new strategy for teaching, assessment and feedback.

Curriculum developers

As for most teachers, textbooks are the curriculum, and in any case, most of them do not have facilities for giving handouts. It is imperative to provide for research-based practices such as spaced practice, interleaving, activities with manipulatives, tables, questions on deep understanding of concepts, higher-order and metacognitive thinking in learning objectives, textbooks and assessment tools.

Researchers

- There is a need to do research in India on maths education, especially for strategies whose meta-analyses have shown promise.
- Research studies on the effectiveness of the research-based teaching strategies in India, e.g., retrieval, spaced practice, interleaving, active learning, discussion, formative evaluation, feedback, and collaborative learning.
- Strategies for development and assessment for deep understanding of concepts and higher-order thinking.
- Research on characteristics of teacher training and PD of teachers that are effective for improving math education.

- Meta-analyses of studies done in India on the effectiveness of different approaches for which many studies are available for improving maths education, e.g., interventions targeting foundational skills and technology integration.

- Explore the longitudinal impact of different interventions on maths learning at the elementary and secondary levels, teacher training, and PD of teachers.

- National Educational Policy (2020) recommends integrating Vedic Maths in maths education. There is a need to do research on its effectiveness for various objectives of maths education.

6. Final Reflections

Effective math pedagogy is not just about improving test scores—it is also about developing problem solving, critical thinking, growth mindset, perseverance and joy in learning. This book advocates teaching evidence-based strategies and research rather than fads and ideology. By aligning evidence with practice, India can unlock the mathematical potential of every learner.

Let us commit to a vision of mathematics education where every student feels capable, every teacher feels supported, and every classroom becomes a space of inquiry and wonder. Access to digital technologies will be a great help as they

- Support personalised, interactive, and inclusive learning and engagement of students.

- Empower teachers.

- Help overcome traditional barriers.

Glossary

Action research. The systematic implementation of an instructional strategy, followed by the collection and analysis of data to determine the effect of that strategy on learning.

Amygdala. The almond-shaped structure in the brain's limbic system that encodes emotional messages for long-term storage.

Angular Gyrus. It is a part of the parietal lobe, near the temporal lobe. It is involved in language-related aspects of math, such as understanding math symbols, word problems, and mathematical terminology.

Anterior Cingulate Cortex (ACC). It is a part of the limbic system, near the frontal lobe, and does error detection and conflict monitoring during maths tasks. It helps in regulating attention and emotional responses to challenging problems.

Auditory Cortex – A part of the temporal lobe that processes sounds, including pitch, volume, rhythm, and speech, and enables us to understand and interpret what we hear.

Axon. The neuron’s long and unbranched fibre that carries impulses from one cell to the neighbouring neurons.

Basal Ganglia. It is located deep in the cerebral hemispheres. It is involved in procedural learning and automaticity, e.g. number facts (addition and multiplication of one-digit numbers and subtraction and division of their sum or product by one-digit numbers).

Brain stem. A major part of the brain that receives sensory input and monitors vital functions such as heartbeat, body temperature, sleeping and digestion.

Broca's area. It is a region in the left frontal lobe of the brain involved in speech production and comprehension.

Cell body: It contains the nucleus and is responsible for maintaining cell function.

Cerebellum. A major part of the brain, located in the rear above the brainstem, that is largely responsible for coordinating muscle movement.

Cerebrum. The largest of the major parts of the brain, it controls sensory interpretation, thinking, and memory.

Chunking. The ability of the brain to perceive a coherent group of items as a single item or chunk, thus increasing the capacity of working memory.

Cognitive overload. A condition where the flow of information exceeds the brain's ability to process and store it.

Collaborative learning: An educational practice through which two or more people collaborate to solve a problem. Roles may or may not be assigned to different students.

Concept mapping: The creation of graphic representations of course content by arranging new information about information they have already mastered.

Cortex. The outer layer of the brain that controls higher mental functions such as thinking, memory, language, and voluntary movement. It is made up of grey matter and is divided into areas that process sensory information, plan actions, and support reasoning and problem-solving. It contains all the neurons.

CRA approach. It involves beginning instruction with hands-on activities and manipulatives to provide a concrete foundation for understanding and gradually transitioning from concrete experiences to pictorial representations and finally to words and symbols.

Critical thinking: A set of cognitive skills—including analysis, inference, evaluation, interpretation, deductive reasoning, and inductive reasoning—that collectively enable a thinker to reasonably decide who and what to believe.

Consolidation. The process by which information is transformed into a mental representation that can be stored in the brain. In learning mathematics, encoding involves making sense of new information—such as understanding a concept, recognizing patterns, or linking ideas to prior knowledge—so it can later be recalled and applied.

Curiosity: It relates to the urge to explain the unexpected, to resolve uncertainty, or the urge to know more.

Declarative neural pathway. This pathway uses working memory and the hippocampus to interface with long-term memory. It is mostly conscious and flexible.

Dendrite. The branched extension from the cell body of a neuron that receives impulses from nearby neurons through synaptic contacts.

Dopamine: It is a hormone that is involved in reward, motivation, and reinforcement of learning.

Dyscalculia. It is a learning disability that results in difficulty in learning mathematical facts and manipulation of numbers and understanding numbers.

Dyslexia. It is a learning disability which causes a persistent problem in learning to read.

Encoding. The process by which newly learned information becomes stable and long-lasting in memory. Consolidation often occurs during rest or sleep, when the brain strengthens and integrates mathematical knowledge and problem-solving strategies, making future recall and application easier.

Exit ticket. They are typically a small piece of paper or an index card on which students respond to a prompt from the teacher, which may involve retrieval, self-reflection, how a concept is related to a concept learned earlier, or application of a concept at the end of a lesson.

Explicit teaching strategies: Explicit instruction is characterised by a series of supports or scaffolds whereby students are guided through the learning

process with clear statements about the purpose and rationale for learning the new skill, clear explanations and demonstrations of the instructional target, and guided practice with feedback until independent mastery of a task has been achieved.

Fixed mindset. A fixed mindset is the belief that you are born with certain innate abilities which cannot be altered.

Frontal lobe. The front part of the brain that monitors higher-order thinking, directs problem-solving, and regulates the emotional (limbic) system.

Glial Cells: These support and protect neurons.

Grey matter. The thin, tough covering of the brain's cerebrum, also known as the cerebral cortex.

Growth mindset: A growth mindset is the belief that abilities can be developed through effort and persistence.

Gyrus. It is a ridge on the cerebral cortex that increases its surface area and function.

Hypothalamus. A small brain structure at the base of the limbic area that regulates body functions in response to internal stimuli.

Hippocampus. A brain structure that encodes information from working memory into long-term memory.

Hormones. These are chemicals released into the bloodstream that influence brain function, e.g. cortisol, which affects memory and stress responses, oxytocin, which influences social bonding and trust.

Inductive reasoning: It involves deriving a rule or a law from several examples.

Interleaving: It involves learning or practising multiple related skills and topics simultaneously while alternating between them.

Journaling. Journaling in mathematics is a reflective practice where students write about their thought processes, problem-solving strategies, observations, and insights related to mathematical concepts.

Laterality of the brain. The notion that two hemispheres of the brain are specialised and process information differently.

Limbic area. The structure at the base of the cerebrum that processes and controls emotions.

Long-term memory. Long-term memory is a nearly permanent storage space in the neocerebrum, where memories and experiences are stored permanently.

Low-floor and high-ceiling tasks. Tasks that are accessible to less proficient students (low-floor) while providing opportunities for deep and extended thinking (high-ceiling) for advanced students.

Maths Anxiety. It is a negative emotional response to learning mathematics**.**

Meta-cognition. Metacognition is thinking about thinking. It involves monitoring one's learning, reflecting on it and adjusting one's learning strategies based on it.

Mind-set. The beliefs one has about one's intelligence, talents, and personality.

Mnemonics. A word or phrase used as a device for remembering unrelated information, e.g. SOH, CAH, TOA for the definitions of sine, cosine and tangent of an angle in a right angled triangle.

Motivation. It is a drive that influences actions and behaviours.

Myelin. A fatty substance that surrounds and insulates a neuron's axon.

Myelination. It is the formation and development of myelin sheaths in the nervous system. This sheath allows nerve signals to travel faster and more efficiently

Neocortex – It is the largest and most recently developed part of the cerebral cortex and is made up of six layers of neurons. It is involved in advanced mental abilities such as reasoning, planning, language, and conscious thought and is especially well developed in humans.

Neuron. The basic cell making up the brain and nervous system, consisting of a cell body, a long fibre (axon) that transmits impulses, and many short fibres (dendrites) that receive them.

Neuroplasticity. It refers to the brain's remarkable ability to reorganise its structure, functions, and neural connections in response to learning, experience, injury, or environmental changes.

Neurotransmitter. One of several dozen chemicals stored in the axon sacs that transmit impulses from neuron to neuron across the synaptic gap.

Neurotrophins. These are proteins that support neuron growth and survival; e.g., brain-derived neurotrophic factor (BDNF) is crucial for synaptic plasticity and learning.

Nodes of Ranvier. These are located between sheaths of myelin. They facilitate speedy electric transmission.

Number sense: Number sense refers to a person's intuitive understanding of numbers, ability to subitise (recognising numbers presented visually without counting), seeing relationships between numbers, flexibility with numbers, and ability to decompose, recompose, and manipulate numbers in various ways, making reasonable guesses about quantities or calculations, estimating and judging the reasonableness of answers.

Occipital lobe. The rear part of the brain which is mainly responsible for visual processing.

Parietal Lobe. The parietal lobe is a region of the brainthat plays a key role in processing sensory information such as touch, temperature, and spatial awareness. It helps with understanding numbers, performing calculations, judging quantity, and visualizing spatial relationships.

Peer assessment. The use of students to provide assessment of their peers' work according to a rubric.

Persistence. It is the ability to persist when you face challenges, setbacks, or confusion.

Prefrontal cortex. The area of the brain just behind the forehead that controls the planning, decision-making, reasoning, and execution of behaviour and that integrates information in working memory.

Primacy-recency effect. The phenomenon whereby one tends to remember best what comes first in a learning episode and second best what comes last.

Procedural neural pathway. This neural pathway primarily uses the basal ganglia to interface with long-term memory. It also integrates with the cerebellum. It is largely unconscious and not as flexible as the declarative pathway.

Procrastination. It is the voluntary and irrational delay of working on a task despite expecting the negative consequences of that delay.

Professional development. It refers to educators actively improving their knowledge, skills, and competencies through learning opportunities to become more effective in teaching by staying updated by deepening understanding of content and recent research on pedagogy.

Retrieval. It involves bringing information to mind from your memory.

Scaffolding. It is an educational practice by which a teacher first provides outside assistance for doing a task and then gradually removes it for students to complete educational tasks independently.

Schwann cells. These are found in the peripheral nervous system. They play a key role in helping nerves to work efficiently by wrapping around nerve fibres and forming a protective layer called the myelin sheath. This insulation allows electrical signals to travel quickly and smoothly along the nerves.

Self-efficacy: It refers to a student's belief in his ability to succeed in maths tasks and challenges.

Short-term memory. A temporary storage space, where information is stored for a short time (some seconds) before its transfer into long-term memory,

Soma: It contains the nucleus and is responsible for maintaining cell function. Also called the cell body.

Spaced practice. The repetition of a skill over increasingly longer periods to improve performance.

Spatial reasoning. It refers to the ability to visualise objects and their movements in the mind, understand relationships between objects (e.g. distance, direction, and orientation), manipulate objects mentally or physically (e.g. rotating, flipping, or transforming shapes), interpret and create visual representations (e.g. maps, graphs, and diagrams) and perceive how they would look when viewed from different sides.

Subitising. The ability to determine the number of objects at a glance without counting.

Sulcus. A shallow groove that surrounds the gyrus.

Synapse. The microscopic gap between the axon of one neuron and the dendrite of another. Neurons communicate via neurotransmitters here.

Synaptic density. It is the number of synapses in a given volume of brain tissue. It is a key indicator of brain development, plasticity and overall cognitive function.

Temporal lobe. The region on each side of the brain which is responsible for memory and auditory processing.

Thalamus. A part of the limbic system that receives all incoming sensory information except smell and shunts it to other areas of the cortex for additional processing.

Visual Cortex – A part of the occipital lobe that receives and interprets information from the eyes. It helps us recognise shapes, colours, movement, and spatial patterns, making visual perception possible

Wernicke area. The region of the brain, usually located in the left hemisphere, is thought to be responsible for sense and meaning in one's native language.

White matter. The support tissue that lies beneath the brain’s grey matter.

Working memory. Working memory is a site in which a person holds and manipulates information for a brief period of time, typically because this information is relevant to a current task.

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Zhang, L., Shang, Y., Huang, Y., & Li, X. (2019). The effects of mathematics anxiety on mathematics achievement: A meta-analysis. *Frontiers in Psychology*, 10, 2903. https://doi.org/10.3389/fpsyg.2019.02903

Rutherford, A., (2021). *Mathematical thinking – for people who hate math; levelup analytical and creative skills. excel at problem solving and decision making.* ARB Publications.

Willis, M.D. (2010). *Learning to Love Math: Teaching Strategies That Change Student Attitudes and Get Results.* ASCD,

Resources

Further Readings

I. Insights from Cognitive Psychology for Learning

1. Agarwal, P. K., & Bain, P. M. (2019). *Powerful Teaching: Unleash the Science of Learning*. Jossey-Bass.
   This book connects cognitive science with classroom practice, presenting strategies like retrieval practice, spacing, and metacognitive feedback that significantly enhance student retention and engagement. Gives evidence for effectiveness of these in enhancing achievement when implemented in classroom settings

2. Brown, P. C., Roediger, H. L. III, & McDaniel, M. A. (2014). *Make It Stick: The Science of Successful Learning*. Belknap Press of Harvard University Press.
   Explains learning strategies such as retrieval, interleaving, spaced repetition, elaboration, generating and reflecting. Gives research evidence for effectivenessof these learning strategies in classroom settings.

II. How the Brain Learns Mathematics

1. Knightly, R. (2023). *The Neuroscience of Learning.* BlueNome Publishing

   Ryan Knightly combines the latest neuroscience findings with practical advice to help readers improve their cognitive abilities. The book covers a wide range of topics related to how the brain learns, the role of neurotransmitters, the function of neural networks, and the science behind memory formation and retention. It also explores the impact of factors like emotion, sleep, nutrition, and exercise on brain performance and learning outcomes. Knightly aims to provide a transformative guide that goes beyond simple education theory,

offering tools and techniques for creating a growth mindset and developing lifelong learning skills.

2. Sousa, D.A. (2014). *How the Brain Learns Mathematics* (2nd ed.). Corwin.
   The second edition of Sousa's foundational text translates emerging neuroscientific research into practical strategies for mathematics educators, focusing on the cognitive development of mathematical reasoning from a young age. It delves into how the brain processes numerical concepts, establishes foundational number sense, and develops memory pathways for mathematical facts and procedures. This edition emphasises the distinction between fact-based, procedural, and conceptual learning and provides educators with brain-compatible approaches to design lessons that enhance retention and build a deeper, more flexible understanding of mathematics rather than just rote memorisation.

3. Oakley, B., Rogowsky, B., & Sejnowski, T.J. (2021). *Uncommon Sense Teaching: Practical Insights in Brain Science to Help Students Learn.*

   This book is a practical guide for educators, applying the latest brain science directly to classroom teaching strategies. It builds on the concepts from *Learning How to Learn* and tailors it for teachers. The book explains the roles of working memory, long-term memory, and cognitive load in student learning, providing concrete methods to help students retain information and develop fluency. It offers strategies for reaching diverse learners, managing a classroom effectively, and designing lessons that align with how the brain naturally acquires and processes information, empowering teachers to make their instruction more inclusive and effective.

   For a summary, watch https://youtu.be/ajxFTsI9tRU

III. Teaching and Learning Strategies for Mathematics

1. National Council of Teachers of Mathematics (2014). *Principles to Actions: Ensuring Mathematical Success for All*. NCTM.

   This book offers guidance to teachers, mathematics coaches, administrators, parents, and policymakers by providing a research-based description of eight essential mathematics teaching practices and the conditions, structures, and policies that must support them.

2. Leinwand, S. (2009). *Accessible mathematics: 10 instructional shifts that raise student achievement.* Heinemann.

   Leinwand presents ten powerful, instructional shifts designed to make math instruction more transparent and effective in raising student achievement. The book focuses on practical, immediately implementable strategies—such as using multiple representations, embedding vocabulary building, and making "why?" a classroom mantra—that help demystify mathematics for students and foster a classroom culture where all learners can engage with and succeed in math.

3. Lang-Raad, N. D., & Marzano, R. J. (2019). *The new art and science of teaching mathematics.* Solution Tree Press.

   This volume applies Marzano's comprehensive instructional framework specifically to mathematics education, outlining a detailed model for effective teaching built on three domains: feedback, content, and context. It provides math teachers with over 280 strategies to make data-informed instructional decisions, design clear learning goals, engage students, and develop rigorous lessons that promote deep conceptual understanding and problem-solving skills.

4. Boaler, J. (2022). *Mathematical Mindsets: Unleashing Students' Potential through Creative Mathematics, Inspiring Messages and Innovative Teaching* 2nd Edition, Jossey-Bass

   This book: (1) Challenges the myth that some people are "math people" and others aren’t. (2) Shows how a growth mindset (belief that intelligence can grow) improves learning. (3) Highlights the damaging

effects of tracking, timed tests, and fixed-ability labelling. (4) Offers strategies for making math visual, creative, and collaborative (e.g., using open-ended tasks, encouraging mistakes as learning opportunities). (5) Gives classroom examples and research on neuroplasticity to bring about equitable grouping in classrooms.

5. Munson, J. & Williams, C. (2017–2022), *Mindset Mathematics Series – A collection of grade-specific books (K–8) offering activities designed to engage students in open, creative, and visual math tasks that promote deep understanding.*

   The books in “Mindset Mathematics" are designed around the principle of active student engagement with tasks that reflect the latest brain science on learning, viz. (1) There is no such thing as a math person – anyone can learn mathematics to high levels. (2) Mistakes, struggle and challenge are the most important times for brain growth. These are expected to change students’ relationship with mathematics and start them believing in their potential.

   The books provide a collection of low-floor, high-ceiling tasks that are open, creative, and visual. These engage students and help them understand the connections between mathematics concepts by looking at the big ideas at each grade level through visualisation, play, and investigation.

   Liljedahl, P. (2020). *Building Thinking Classrooms in Mathematics, Grades K-12: 14 Teaching Practices for Enhancing Learning.* Corwin.

   Based on over a decade of classroom research, this book outlines 14 evidence-based teaching practices that foster deep mathematical thinking. Strategies include creating visibly random groups, using vertical non-permanent surfaces, and selecting rich tasks to promote engagement, collaboration, and problem-solving.

6. Liljedahl, P., & Giroux, M. (2024). *Mathematics Tasks for the Thinking Classroom, Grades K–5*. Corwin Press.

A companion to *Building Thinking Classrooms*, this resource offers a curated collection of rich mathematical tasks designed to spark curiosity and sustained thinking in young learners. Includes ready-to-use problems, implementation tips, and strategies for differentiation.

7. Parrish, S. (2018). *Number Talks: Common Core Edition, Grades K–5*. Math Solutions.

   Offers strategies for conducting short daily discussions that develop number sense and mental math in elementary students, with a focus on student reasoning and flexible thinking.

8. Humpherys, C. & Parker, P. (2015). Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding Grades 3-10.

   This book provides guidance on how to facilitate short, daily conversations that help students develop computational fluency, number sense, and a willingness to share and critique strategies. It's packed with examples and student dialogues.

9. Gaskin, D. (2022). *312 Things to Do With a Math Journal.* Table Top Academy.

   This is a practical resource book for educators and parents focused on using math journals to deepen students' mathematical understanding. It moves beyond traditional drills by providing over 300 prompts and activities designed to make students think, reason, and communicate their mathematical ideas. The book advocates for journaling as a tool to develop problem-solving skills, conceptual understanding, and mathematical creativity. Activities encourage students to draw pictures, explain their thinking, notice patterns, and pose their own problems, transforming math from a subject of right answers into a dynamic language for exploring ideas.

10. Polya, G. (1945). *How to Solve It*. Princeton University Press.

A foundational book on problem solving in mathematics. Introduces heuristic methods for tackling unfamiliar problems—understanding, planning, executing, and reflecting.

11. Andreescu, T., Cordeiro, K., & Andreescu, A. (2020). *Awesome Math: Teaching Mathematics with Problem-Based Learning*. Jossey-Bass.

    Provides guidance and rich problems to promote deep mathematical reasoning, creativity, and collaboration through a problem-based learning approach for advanced learners.

12. Barton, C. (2018). *How I wish I'd taught maths: Lessons learned from research, conversations with experts, and 12 years of mistakes*. John Catt Educational Ltd.

    This book is about shifting from a teacher-centric model of "How can I make this lesson exciting?" to a learning-centric model of "How can I design instruction that aligns with how students actually learn and remember using principles from cognitive science, such as using worked examples, managing cognitive load, and leveraging formative assessment?"

13. Barton, C. (2020). *Reflect, expect, check, explain: Sequences and behaviour to enable mathematical thinking in the classroom*. John Catt Educational Ltd.

    The second book builds on Barton, C. (2018) by providing a practical framework—the RECE model (Reflect, Expect, Check, Explain)—for sequencing curriculum and crafting lessons that build fluency, anticipate misconceptions, and develop deeper mathematical reasoning through intelligent practice and purposeful questioning.

14. Nathan, D. L.-D. (2025). *Mathematics Teaching Reimagined: Seven Competencies to Foster Student Learning and Engagement.*

    This book presents a forward-thinking framework for effective teaching. Nathan identifies seven essential competencies that modern

educators need to cultivate in order to move beyond traditional instructional methods. The book serves as a guide for reimagining classroom practice to foster greater student engagement and deeper, more meaningful learning. It emphasises a shift in the teacher's role toward fostering a dynamic and interactive learning environment where students are actively involved in constructing their mathematical understanding.

15. Willis, M.D. (2010). *Learning to Love Math: Teaching Strategies That Change Student Attitudes and Get Results.*

    This book addresses the critical link between student emotion and academic achievement in mathematics. Willis explores teaching strategies rooted in brain-based research that are designed to reduce anxiety and build positive attitudes toward math. By creating a classroom environment that fosters confidence and resilience, the book provides educators with practical methods to change students' mindsets from negative to engaged, ultimately leading to improved learning outcomes and better results. The core premise is that by cultivating a love for math, deeper understanding and higher performance will naturally follow.

16. Rutherford, A., (2021). *Mathematical Thinking – For People Who Hate Math; Level Up Analytical and Creative Skills. Excel at Problem Solving and Decision Making.* ARB Publications.

    This book targets individuals who have a strong aversion to mathematics, aiming to reframe it as a practical tool for everyday life rather than an abstract academic exercise. Rutherford focuses on developing the reader's core analytical and creative reasoning abilities, arguing that mathematical thinking is essential for enhancing problem-solving and decision-making skills in both personal and professional contexts. The goal is to help readers "level up" their logical capabilities without getting bogged down by complex calculations, making the power of mathematical thought accessible to

everyone, regardless of their background or initial dislike for the subject.

17. Oakley, B., Sejnowski, T.J., & McConville, A. (2018). *Learning How to Learn: How to Succeed in School Without Spending All Your Time Studying: A Guide for Kids and Teens*.

    This book translates fundamental concepts of neuroscience and cognitive psychology into an accessible and engaging guide for young learners. It explains how the brain learns through two key modes: the focused mode (for intense concentration) and the diffuse mode (for unconscious background processing), emphasising the importance of breaks and sleep for solidifying learning. The book provides practical techniques to overcome procrastination, improve memory using metaphors and the "memory palace" technique, and study more effectively. Its core message is that understanding how your brain works allows you to learn more efficiently and deeply without resorting to ineffective, last-minute cramming.

IV. Teaching for Equity

1. Clement, L., & Bordeaux, C. (Eds.). (2017). Access and equity: Promoting high-quality mathematics in grades 6–8. National Council of Teachers of Mathematics. NCTM.

    It provides specific strategies for ensuring all students, particularly those from historically marginalised populations, have access to high-quality math instruction. It addresses issues of identity, discourse, and equitable teaching practices head-on. Specific techniques for encouraging and supporting participation from all students, not just the quickest and loudest. This includes strategies for wait time, asking probing questions, and orchestrating group discussions. And discusses how to select and adapt mathematical tasks that are "low-floor, high-ceiling" (accessible to all but with room for deep challenge) and that allow for multiple entry points and solution strategies.

V. Formative Assessment

1. National Council of Teachers of Mathematics. (2013). *Formative Assessment in the Mathematics Classroom: Issues and Challenges* (2013 Yearbook). NCTM.

   A collection of research and practitioner perspectives on the theory and practice of formative assessment in mathematics. Addresses challenges in implementation, highlights classroom-based examples, and explores how assessment can be used to inform instruction and improve student learning.

2. Wiliam, D. (2011). *Embedded Formative Assessment*. Solution Tree Press.

   Offers actionable strategies for using student evidence to adapt teaching in real time, with a strong focus on feedback and questioning.

   Provides insights, case studies, and research on the challenges and opportunities of formative assessment in math classrooms.

VI. Teacher Training and Professional Development of In-Service Teachers

1. Nolan, E.C. Dixon, J.K. et al. (2016). *Making Sense of Mathematics for Teaching Series* is a collection of 4 professional development books and resources designed to help teachers of classes (K-2, 3-5, 6-8, and high school) to deepen their own conceptual understanding of mathematics to improve instruction.

   The common themes across all books are conceptual understanding before teaching procedures, use of precise language, use of visual models and representations, development and connection of ideas across grades, providing opportunities to students to grapple with problems, make conjectures, and learn from mistakes, and provision of strategies for formative assessment, e.g., how to listen to student

thinking, interpret their work, and use that information to guide instruction.

2. Boston, M.D., Candela, A.G., & Dixon, J.K. (2019). *Making Sense of Mathematics for Teaching to Inform Instructional Quality.* Solution Tree | Press.

   This book provides a tool kit for improving the quality of instruction in maths. It highlights the use of tasks, questions, and evidence to improve instruction. They emphasise how teachers can select and implement high-cognitive-demand tasks, promote meaningful mathematical discussions, and interpret students' thinking for a coherent, equitable, and high-quality mathematics learning experiences for all students.

3. Darling-Hammond, L., Wei, C. & Andree, A. (2017). *Effective Teacher Professional Development.* CA: Learning Policy Institute

   This book synthesises research to identify the core structural features of high-quality teacher professional development (PD). It argues that effective PD is not a one-day workshop but a sustained, collaborative process focused on content. The authors outline seven key characteristics: it must be content-focused, incorporate active learning strategies for teachers, be coherent with school curricula and policies, of sufficient duration, be collaborative, provide expert support and coaching, and offer opportunities for feedback and reflection. The central conclusion is that for PD to successfully improve teaching practices and student outcomes, it must treat teachers as active learners in a professional community, mirroring the kind of deep, engaged learning we want for students.

4. Loucks-Horsley, S., et al. (2010). *Designing Professional Development for Teachers of Science and Mathematics.*

   This book provides a practical, iterative framework for designing and implementing effective PD, specifically for STEM educators.

Rejecting a one-size-fits-all approach, the authors guide planners through a deliberate process that begins by deeply understanding the context, the needs of the teacher learners, and the content to be taught. The core of the framework involves selecting and combining a variety of PD strategies—such as study groups or action research—based on specific learning goals, emphasising that effective design is a conscious and context-driven decision-making process rather than simply choosing a pre-packaged workshop format.

5. Asian Development Bank (2021). *Teacher Professional Development Case Studies: K-12, TVET and Tertiary Education.* Asian Development Bank.

   This report examines real-world applications of teacher PD across the diverse contexts of Asia and the Pacific through a series of detailed case studies. It highlights that successful PD initiatives are those that are strategically adapted to local needs, often leveraging technology to scale up reach and provide sustainable support, particularly in remote areas. Its overarching conclusion is that investing in high-quality, contextualised PD is a cornerstone of educational development, but it requires long-term commitment, integration with broader systemic reforms, and addressing common challenges like funding, facilitator capacity, and resistance to change to achieve lasting impact.

The schools should buy such books. If it is not possible due to paucity of resources, the Open National Library should get enough copies for teachers to borrow.
Teachers can also go to YouTube to look for videos by these authors.

Online Courses on Coursera

1. **Making Math Click: Understand Math Without Fear**

2. **Uncommon Sense Teaching**

3. **Uncommon Sense Teaching: Part 2, Building Community and Habits of Learning**
4. **Learning How to Learn: Powerful mental tools to help you master tough subjects**
5. **Introduction to Mathematical Thinking**

Websites

1. nctm.org
2. https://illustrativemathematics.org/
3. Mathematics Framework
4. https://tinyurl.com/24n8mkr3
5. https://tipsforteachers.co.uk/
6. https://www.youcubed.org
7. https://nrich.maths.org/
8. Rich Tasks - maths For Love
9. https://www.openmiddle.com
10. Teacher Resources | Alicia Burdess | Grande Prairie Alberta
11. https://www.learner.org/

Appendix 1

Facilities in Indian Schools (UDISE+ 021-2022)

Facility	Percentage of schools that have it	Range in Percentage in different States/UTs
Library/book bank/book reading corner	87.3	23.8 - 100
Number of books available	713	95 - 8524
Computer	47.5	18.3 - 100
Functional computer	45.8	16.3 - 100
Internet	33.9	7.9 - 100
Functional desktops/pcs	12.9	2.7 - 89
Functional tablets	9.0	0.5 - 66.5
Functional projectors	16.7	3.2 - 84.2
Functional digital boards/smart boards/virtual classrooms/smart TV	14.9	0.0 - 99.9
Functional mobile phones for teaching	17.7	0.2 - 99.9
Digital library	2.2	0 - 10.3
ICT labs	27.8	0 - 91.3
Functional ICT labs	21.0	0 - 87.5

Appendix 2: Effect Size and Meta-Analysis

Effect size

Effect size is a quantitative measure that describes the magnitude of a phenomenon, such as the strength of the relationship between variables or the difference between means of two groups exposed to different treatments. Unlike p-values, which only indicate whether an effect exists (statistical significance), effect sizes provide information about the practical importance of the findings. Furthermore, effect sizes from many similar studies can be combined using statistical techniques to find the average effect size and its confidence interval.

For two samples or variables with which most of our studies cited in this book are concerned, it is generally measured by Cohen's d, Hegde's g or Pearson r.

Cohen's d (for independent samples), is given by

$$d = \frac{\overline{X}_1 - \overline{X}_2}{s_{pooled}}$$

Where $\overline{X}_1$ and $\overline{X}_2$ are sample means of the two groups,

s_{pooled} is the pooled standard deviation, calculated as:

$$s_{pooled} = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$$

Where n_1 and n_2 are the sample sizes

s_1 and s_2 are the sample standard deviations.

If sample sizes are small, Hedges' g (a bias-corrected version of Cohen's d) by applying a correction factor may be used. The Hegde's g is given by:

$$g = d \times \left(1 - \frac{3}{4(n_1+n_2)-9}\right)$$

Hedges' g is preferable when sample sizes are small (n < 20) per group and in meta-analyses.

If results from correlational and experimental studies are to be combined for finding the overall effect size. Then r can be converted into effect size d by

$$d = \frac{2r}{\sqrt{1-r^2}}$$

Interpretation of Cohen's d and Hedges' g

These are generally interpreted as follows:

Small: ~ 0.2

Medium: ~ 0.5

Large: ~ 0.8

Pearson's r is interpreted as follows:

Small: ~ 0.1

Medium: ~ 0.3

Large: ~ 0.5

However, there are no universal thresholds for "small", "medium", or "large". What's "large" in one field may be "small" in another. Even very small effect sizes in some studies may be practically important, e.g., in the medical field, where a treatment whose effect size, even though small, may save millions of lives.

Effect size helps to avoid overreliance on p-values (which depend on sample size), determine sample size for a study to have high power, and is essential for meta-analyses (combining results from multiple studies).

Meta Analyses

Meta-analysis is a statistical method used to systematically combine and analyse the results of multiple independent studies on the same topic to draw an overall conclusion. It provides a more precise and reliable estimate of an effect (e.g. treatment effectiveness, relationship between variables) than any single study alone.

Unlike a traditional literature review (which is narrative), meta-analysis uses statistical techniques to pool data. In pooling data, larger studies (with more participants) are given more weight because they have greater precision.

Meta-analysis involves first checking if differences between studies are due to random variation (homogeneous) or true differences (heterogeneous) by using statistical tests to decide between a fixed and random model. And secondly if there is publication bias, that is an assessment of whether small, non-significant studies are missing.

Meta-analysis has more power (probability of rejecting the null hypothesis if false) and gives narrower confidence intervals than individual studies. Furthermore, it can resolve conflicts by using moderator variables.

However, poor-quality studies and combining very different studies can lead to misleading results.

Interpretation of improvement of scores in standard deviation.

Some meta-analyses are reported in terms of improvement of standard deviation (SD) in math scores. Researchers often approximate 1 SD ≈ 1 year of learning progress in school (though this varies). Thus, an improvement of 0.12 SD would mean that, on average, students performed about one-eighth of a standard deviation better — roughly equivalent to 1–2 months of additional learning progress.

Confidence Interval (CI) for average or weighted average effect size in meta-analysis.

CI for average or weighted average effect size refers to the range within which the intervention effect is likely to lie, indicating a magnitude of uncertainty around mean effect size. A short confidence interval states a low uncertainty and precise estimate, while a broader interval depicts higher uncertainty, implying the need for more data.

Fixed-Effect vs. Random-Effect Models for Combining Effect Sizes

When conducting a meta-analysis, researchers must choose between fixed-effect and random-effects models to estimate the overall effect size. The choice depends on assumptions about the underlying data and the goals of the analysis.

Fixed-effect models assume that all studies share a single, common true effect size; any observed differences in effect sizes are due solely to sampling error (within-study variability). It can be used when studies are highly similar (e.g. identical methods, populations, and interventions). However, it is rarely justified in practice because true effect sizes usually vary across studies.

Random-Effects Models assume that studies estimate different true effect sizes (due to methodological, population, or intervention differences) and observed variability comes from both sampling error and true between-study differences. These assumptions account for the heterogeneity of the between-study variance.

This model is almost always preferred in real-world meta-analyses because effects vary across studies.

It should also be used when studies differ in design, settings, or populations.

Moderator Variables in Effect Size Analysis

A moderator is a third variable that changes how the independent variable affects the dependent variable.

It "modifies" the effect size, making it larger, smaller, or even reversing it under different conditions.

It identifies when/for whom an effect holds, explains heterogeneity (variability in effect sizes across studies) and generalises findings (e.g. developing growth mindset is more effective in improving achievement of less proficient students than for high-proficient students)

Moderator variables are critical for understanding context-dependent effects in research. By analysing moderators, researchers can know if an effect applies only in certain conditions and improve interventions and resolve conflicting findings across studies.

Robustness of meta-analysis

It is a qualitative judgment based on a combination of quantitative and qualitative criteria viz., the number of studies and total sample size (the "N"); the quality of the included primary studies; the homogeneity of the results; the presence and analysis of moderator variables; and assessment of publication bias

Appendix 3: Examples of Application of Different Problem Solving Strategies

There are many strategies that can be used for solving problems. We illustrate the use of those strategies for solving problems.

All these problems can be solved by using other strategies too.

1. Use hands-on objects or role-play the situation

Ask students to divide 10 objects into two groups and express it as a number sentence.

Teachers can write the number sentences for different groupings and ask questions

What do you see?

What do you wonder?

What strategy would you use to check that you have not missed any combination if you were asked to divide a specific number of sticks in as many ways as you can?

Give 12 counters or sticks and ask them to divide it equally among 4 students and express it as a number sentence.

2. Guess and Check (with refinement)

It involves making an educated guess, checking if it works, and then using the result to refine your next guess.

Example: A shopkeeper has some 5-kg weights and some 1-kg weights.He has 10 weights, and they weigh 30 kilograms in all. Find the number of 1-kg and 5-kg weights.

You can try different number of the two-weights, and check if they add to 30 kg or not.

If they don’t, revise them, until they add to 30. For example,

Guess the number of weights of		Find Total weight	Is it 30 kg?
5-kg	1-kg		
4	6	4×5+6×1=26	No
6	4	6×5+4×1=34	No
5	5	5×5+5×1=30	Yes

3. Make a Table

Organising information systematically can help reveal relationships or possibilities.

Example: How many different ways can you make 25 rupees using one-rupee, five-rupee and ten-rupee notes?

Create a table to list the combinations:

10-Rupee notes	5-Rupee notes	1-Rupee notes	Total amount
2	1	0	25
2	0	5	25
1	3	0	25
1	2	5	25
1	1	10	25
1	0	15	25
0	5	0	25
0	4	5	25

0	3	10	25
0	2	15	25
0	1	20	25
0	0	25	25

4. Create an organised list

Example: Five players – A, B, C, D & E – enter a badminton tournament in which each player must play every other person exactly once. Determine the total number of games that will be played.

Let us make a list of matches a player can play, beginning with player A

Player	Will play with players	Reason	Number of matches
A	B, C, D, E	Is the first player before the start of the tournament	4
B	C, D, E	As he has already played with A	3
C	D & E	As he has already played with A and B	2
D	E	As he has already played with A, B and C	1

There will be 4 + 3 + 2 + 1 = 10 matches.

5. Simplify the problem

If a problem seems too complex, try solving a simpler version of it first. This can help you identify the core principles or develop a strategy that can then be applied to the original problem.

Example: How many diagonals does a polygon with 20 sides have?

1. Start with the number of diagonals in polygons with small number of sides, and find the number of diagonals in them, e.g.,

   Triangle (3 sides)- 0 diagonals

   Square (4 sides)- 2 diagonals

   Pentagon (5 sides): 5 diagonals

   Hexagon (6 sides): 9 diagonals

2. Look for a pattern or a formula. You might notice that from each vertex of an n-sided polygon, you can draw (n-3) diagonals. Since each diagonal connects two vertices, you divide by 2: n(n−3)/2.

3. Apply the formula for n = 20 - $\frac{20(20-3)}{2}$ = 170.

6. Problem solving with algebra

Example: Taxi A charges a flat fee of 25 INR per km and Taxi B charges INR 100 + INR 20 per km.

(a) Is there a mileage at which the charges would be the same for the two taxis?

(b) For what distances does taxi service B cost less than taxi service A?

(a) Let n be the number of miles for which the taxi fare is the same for both taxi services.

Taxi charges for n km for service A will be INR 25n.

Taxi charges for n km for service B will be INR (100 + 20n).
If charges for n km are the same for both, then

25n = 100 + 20n

Solving for n, gives n = 20 km.

(b) Taxi Service B will be more economical for distances greater than 20
20 km.

Example 2. I am a two-digit number. If you reverse my digits and add to the original number, I am a multiple of 11. Can you explain why?

Let the ten's digit be 'a' and the one's digit be 'b'.

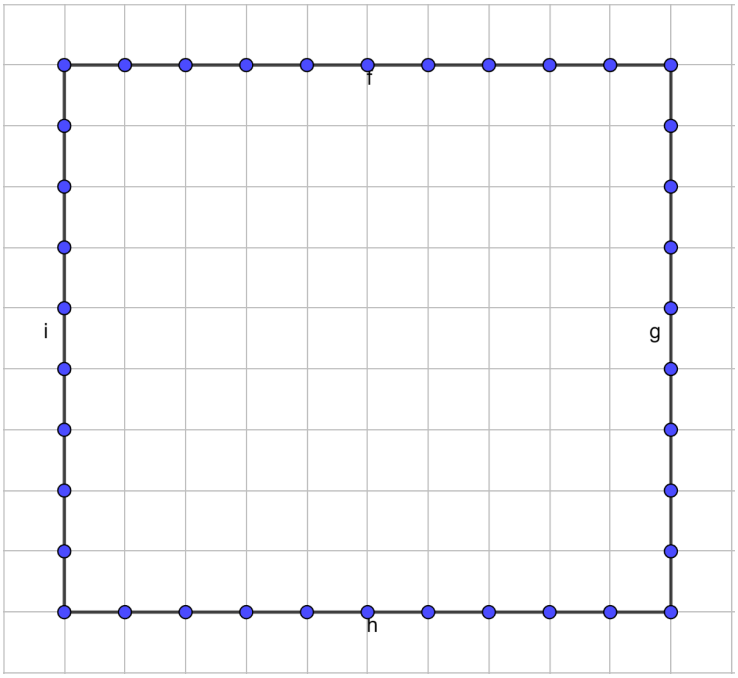
Then the number = 10a + b.

When we reverse the digits, it will be 10b + a.

The sum of the two numbers = 10a + b + 10b + a = 11a + 11b = 11(a + b), which is a multiple of 11.

7. Make a Drawing or Diagram

A farmer has made a hedge around a square that has 10 poles at equal distances on each side. How many poles does he need to use?



Now you can easily see that the number of poles he needs is 36.

8. Work Backwards

This strategy is effective when the end result is known and you need to find the starting point or a previous state.

Example: Alice bought a book for INR 50 and then spent half of her remaining money on a dress. She had INR 100 left. How much money did she start with?

Alice had INR 100 left after buying a shirt. This INR 100 is half of the money she had before buying the dress. So, before buying the dress, she had 2×INR 100 = INR 200.

She had INR 200 after buying the book. She spent INR 50 on the book. So, she started with INR 200 + INR 50 = INR 250

9. Use Logical Reasoning

Strategy: Eliminate impossible answers and use logic.

Example: A number is divisible by 3 and 5 but not by 10. What could it be?

- Must be divisible by LCM of 3 and 5, which is 15
- Not divisible by 10 → eliminate multiples of 15 with even numbers as their multiple with 15 will be a multiple of 10 and hence divisible by 10.
- All odd multiples of 15 – 15, 45, 75, … will be divisible by 3 and 5.

10. Look for a Pattern

Example: Find the sum of first 10 odd numbers

List the sum of a few odd numbers beginning with 1

$1 = 1$

$1 + 3 = 4$

$1 + 3 + 5 = 9$

$1 + 3 + 5 + 7 = 16$

Notice that the sum of the first 1, 2, 3, and 4 odd numbers is equal to the square of these numbers.

Therefore, the sum of 20 odd numbers beginning with 1 will be $20^2 = 400$.

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